Quivers and Koszul duality:

~~derived~~ category $\mathcal{C}$.

~~Fukaya~~

triangulated

(dg-enhanced)

~~(A_∞? $(\infty,1)$?)~~

Some objects in $\mathcal{C}$:

$$\begin{tikzcd} \bullet \arrow[r, bend right, "b"'] & \bullet \arrow[l, bend right, "a"'] & \bullet \arrow[l, "f"'] \arrow[loop right, "x"] \end{tikzcd}$$

(generators for) + all Hom's

$baf = fx$, $fx^2 = 0$

Q: quiver w/ relations (dg) $\longleftrightarrow$ finite category

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[dr, hook] & \mathcal{C} \\ & \langle Q \rangle \arrow[u, phantom, "\cup\!\mathrm{I}"] \end{tikzcd}$$

$\langle Q \rangle$ full tri. subcat. gen'd by Q (cones)

goal:

completing Q (without $\mathcal{C}$):

$$\begin{tikzcd} Q \arrow[r, hook, "\text{Yoneda}", "Y"'] \arrow[dr, hook] & \underset{\text{f.d.}}{\mathrm{Rep}(Q^{op})} = \mathrm{Fun}(Q^{op}, \mathrm{Vect}) = \mathrm{Mod}_{\mathbb{C}Q} \\ & \mathcal{D}^b(\mathrm{Mod}_{\mathbb{C}Q}) \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$\boxed{\mathrm{Perf}}$ $\rightarrow$ claim: This is $\langle Q \rangle$. $(\forall \mathcal{C})$ ☆

What is Y?

$v \in Q \longmapsto P_v \in \mathrm{Rep}(Q^{op})$ vertex projectives

- v.s. at $w = \mathbb{C}^{\oplus \{\text{paths } w \rightsquigarrow v\}}$
- arrows: path composition

$A := \mathbb{C}Q = \{\text{paths}\} = \bigoplus_v \{\text{paths ending at } v\} = \bigoplus_v P_v$.

$P_v = e_v \cdot A$, (e_v empty/identity path at v)

$\mathrm{Hom}(P_v, M) = \{\text{images of } e_v\} = M_v$ (v.s. of M at v, $M \in \mathrm{Rep}(Q^{op})$)

$\Rightarrow \mathrm{Hom}(P_v, P_w) = \{\text{paths } v \rightsquigarrow w\} \Rightarrow Y$ embedding.

digression: $S_v \in Rep(Q^{op})$ "$\mathbb{C}$ at vertex v" vertex simples

$$\cdots \to \bigoplus_{\substack{\text{rel}^n\text{'s on} \\ \text{paths } u \rightsquigarrow v}} P_u \longrightarrow \bigoplus_{\substack{\text{arrows} \\ w \to v}} P_w \longrightarrow P_v \to S_v \to 0$$

$$\Rightarrow \mathrm{Ext}^1(S_v, S_w) = \mathbb{C}^{\oplus\{\text{arrows } w \to v\}}, \quad \mathrm{Ext}^2(S_v, S_w) = \mathbb{C}^{\oplus\{\text{rel}^n\text{'s on paths } w \rightsquigarrow v\}}$$

no relations $\Rightarrow \mathrm{Ext}^{\geq 2}(S_v, S_w) = 0$

$\Rightarrow$ actually gl. dim ≤ 1.

exercise: write down a resolution for any M in this case. (or diagonal)

caveat: loops $\rightsquigarrow$ complications

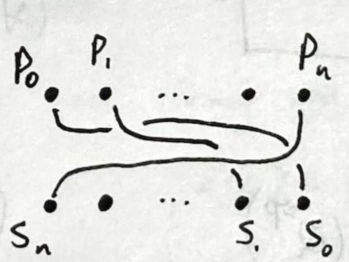
$\mathbb{C} \ni a \quad \forall a \in \mathbb{C}$ simple module

$$\mathrm{Perf}(Q) = \langle P_v \rangle \overset{\text{"often"}}{=} \langle S_v \rangle.$$

$$\langle P_v : v \in I \rangle^{\perp} = \langle S_w : w \notin I \rangle, \quad {}^{\perp}\langle S_w : w \notin I \rangle = \langle P_v : v \in I \rangle$$

$$\mathrm{Hom}(P_v, S_w) = \mathbb{C}^{\delta_{vw}} \quad \text{"dual bases"}$$

Q directed $\Rightarrow$ both F.E.C.'s related by mutation



exercise: what do you get from Beilinson's F.E.C. on $\mathbb{P}^n$?

Back to ★: ...

Back to (✯):

$$\begin{tikzcd} Q \arrow[r, hook, "i"] \arrow[d, hook, "Y"] & \langle Q \rangle \subseteq \mathcal{C} \arrow[dl, dashed, bend left, "\mathrm{Hom}(G,-)"] \\ \mathrm{Perf}(Q) \arrow[ur, dashed, "- \otimes_{\mathbb{C}Q} G"] & \end{tikzcd}$$

$$\underset{\mathrm{Perf}(Q)}{P_v} \longmapsto \underset{\mathcal{C}}{i(v)} \qquad \mathrm{Hom}(G, i(v)) = P_v$$

$$\mathbb{C}Q \longmapsto \bigoplus_v i(v) =: G, \qquad \mathrm{End}(G) = \mathbb{C}Q$$

$\Leftarrow$ Q gen's all Hom's between $i(v)$'s

$$M \longmapsto \underbrace{\underbrace{M}_{\text{cone on } P_v\text{'s}} \otimes_{\mathbb{C}Q} G}_{\text{cone on } i(v)\text{'s}}$$

so $\mathrm{Perf}(Q) \simeq \langle Q \rangle \subseteq \mathcal{C}$

$\quad \mathrm{Perf}(Q) = \mathrm{Perf}(\mathrm{End}(G))$, $\quad \langle Q \rangle = \langle G \rangle$ ■

Koszul duality

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remark: admissibility

= extending $\mathrm{Hom}(G,-)$ to $\mathcal{C}$

$\mathcal{C} \xrightarrow[\mathrm{Hom}(G,-)]{} D^b(Q)$, so admissible iff $D^b(Q) = \mathrm{Perf}(Q)$

i.e. Q smooth.

admissibility gives

weak gen. $\Rightarrow$ strong gen.

$\langle Q \rangle^{\perp} = 0 \qquad \langle Q \rangle = \mathcal{C}$

Morita theory:

change generators

$\rightsquigarrow \mathrm{Perf}(A) \simeq \mathrm{Perf}(B)$

$$\begin{tikzcd} \mathrm{Perf}(A) \arrow[r, bend left, "\mathrm{Hom}_A(M,-)"] & \mathrm{Perf}(B) \arrow[l, bend left, "- \otimes_B M"] \end{tikzcd}$$

generator $M \in \mathrm{Mod}_A$

$\mathrm{End}(M) = B$ i.e. $M \in {}_B\mathrm{Mod}_A$.

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for this to preserve hearts / t-structures

(i.e. $\mathrm{Mod}_A \simeq \mathrm{Mod}_B$)

we want $\mathrm{Hom}_A(M,-)$, $- \otimes_B M$ underived,

i.e. M projective for A and B

$\rightsquigarrow$ classical Morita equivalence

remark: Toen:

"all functors" on dg-cats are bimodules

$=$ FM kernels

examples!: ① $\mathcal{C} = D^b\left(\underset{\underset{\displaystyle R}{\parallel}}{\mathbb{C}[x]/x^2}\right) = \langle R/x \rangle$

$$\underbrace{R/x \xrightarrow{x} R \longrightarrow R/x}_{\theta}$$

$$\mathrm{End}(R/x) = \mathrm{Ext}^*_R(R/x, R/x) = \mathbb{C}[\theta], \ |\theta| = 1$$

$$\Rightarrow D^b(R) \simeq \mathrm{Perf}(\mathbb{C}[\theta]) = D^b(\text{“}\mathbb{A}^1_\theta\text{”}) \quad \text{graded } \mathbb{A}^1.$$

$$\cup$$

$$\langle \mathbb{C}[\theta]/\theta \rangle = D^b_0(\mathbb{A}^1_\theta)$$

↖ supported at origin

$$\mathrm{End}(\mathbb{C}[\theta]/\theta) = \mathbb{C}[x]/x^2 = R$$

$$\Rightarrow D^b_0(\mathbb{A}^1_\theta) \simeq \mathrm{Perf}(R).$$

$$\cap \qquad\qquad \cap$$

$$D^b(\mathbb{A}^1_\theta) \simeq D^b(R)$$

$$\begin{tikzcd} R \arrow[r, squiggly, bend left, "\langle R/x \rangle"] & \mathbb{C}[\theta] \arrow[l, squiggly, bend left, "\langle \mathbb{C}[\theta]/\theta \rangle"] \end{tikzcd}$$

classical Koszul duality.

② $Q = \left(\begin{tikzcd} \bullet_0 \arrow[r, bend left, "q"] & \bullet_1 \arrow[l, bend left, "x"] \arrow[loop right, "t"] \end{tikzcd} \middle/ qx = t^2 - 1 \right)$

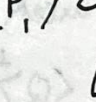
$\mathrm{Perf}(Q) = \langle P_0, P_1 \rangle$

$\mathrm{End}(P_1) = \mathbb{C}[t] : D^b(\mathbb{A}^1_t) \simeq \langle P_1 \rangle \subset \mathrm{Perf}(Q)$

$\cup$ (full) subcategory

$$\mathrm{End}(P_0) = \frac{\mathbb{C}[xq, xtq]}{(xtq)^2 = (xq)^3 + (xq)^2} : \mathrm{Perf}(X) \simeq \langle P_0 \rangle$$

$X : Y^2 = X^3 + X^2$

 $\subset \mathbb{A}^2$

$\mathbb{A}^1_t$ smooth $\Rightarrow \langle P_1 \rangle$ admissible

$\Rightarrow \exists$ S.O.D. $\mathrm{Perf}(Q) = \langle \mathrm{proj}_{P_1^\perp}(P_0), P_1 \rangle$

$$\parallel$$

$$\mathrm{cone}(P_1 \xrightarrow{x} P_0)$$

$$\parallel$$

$$S_0.$$

“nc-resolution of X”

⚠ $\mathrm{Perf}(X)$ not a subcategory of $D^b(\tilde{X})$ ← resolution of X.

$$\mathrm{End}(S_0) = \mathbb{C}, \quad \mathrm{Hom}(S_0, P_1) = \mathbb{C}^2[-1]$$

$$\rightsquigarrow \mathrm{Perf}(Q) \simeq \mathrm{Perf}\left(\begin{tikzcd} \underset{S_0[-1]}{\bullet} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \underset{P_1}{\bullet} \arrow[loop right, "t"] \end{tikzcd} \Big/ \begin{array}{l} tu = u \\ tv = -v \end{array} \right)$$

$$\| \text{ SOD}$$

$$\underbrace{\langle D^b(pt), D^b(\mathbb{A}^1) \rangle}_{\Rightarrow \text{ smooth, so actually an nc-resolution.}}$$