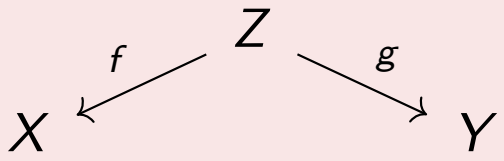
# Categorical resolutions

(an optimistic view)

Calum Crossley

3 Nov 2025

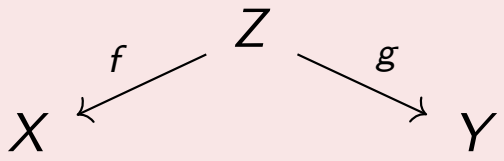
## *DK*-Conjecture (Bondal–Orlov '02, Kawamata '02)

Take a roof 

of smooth projective varieties.

- If $f^*K_X = g^*K_Y$, then $D^b(X) = D^b(Y)$.
- If $f^*K_X > g^*K_Y$, then $D^b(X) = \langle D^b(Y), \ldots \rangle$.

## *DK*-Conjecture (Bondal–Orlov '02, Kawamata '02)

Take a roof 

of smooth projective varieties.

- If $f^*K_X = g^*K_Y$, then $D^b(X) = D^b(Y)$.
- If $f^*K_X > g^*K_Y$, then $D^b(X) = \langle D^b(Y), \ldots \rangle$.

$\Longrightarrow$ MMP passes to smaller and smaller admissible components.

## *DK*-Conjecture (Bondal–Orlov '02, Kawamata '02)

Take a roof $X \xleftarrow{f} Z \xrightarrow{g} Y$ of smooth projective varieties.

- If $f^*K_X = g^*K_Y$, then $D^b(X) = D^b(Y)$.
- If $f^*K_X > g^*K_Y$, then $D^b(X) = \langle D^b(Y), \ldots \rangle$.

$\Longrightarrow$ MMP passes to smaller and smaller admissible components.

## Example (Crepant resolutions should be *D*-unique)

All crepant ($K_{\widetilde{X}} = \pi^*K_X$) resolutions $\pi : \widetilde{X} \to X$ of a Gorenstein singularity are $K$-equivalent $\Longrightarrow$ conjecturally $D$-equivalent.

**$DK$-Conjecture (Bondal–Orlov '02, Kawamata '02)**

Take a roof $X \xleftarrow{f} Z \xrightarrow{g} Y$ of smooth projective varieties.

- If $f^*K_X = g^*K_Y$, then $D^b(X) = D^b(Y)$.
- If $f^*K_X > g^*K_Y$, then $D^b(X) = \langle D^b(Y), \ldots \rangle$.

$\Longrightarrow$ MMP passes to smaller and smaller admissible components.

**Example (Crepant resolutions should be $D$-unique)**

All crepant ($K_{\widetilde{X}} = \pi^*K_X$) resolutions $\pi : \widetilde{X} \to X$ of a Gorenstein singularity are $K$-equivalent $\Longrightarrow$ conjecturally $D$-equivalent.

$\Longrightarrow$ At the level of $D^b(-)$, MMP stops at a *smooth* minimal model. (when a crepant resolution exists)

## Minimal Resolution Conjecture (Bondal–Orlov '02)

- $D^b(X)$ has a minimal "nc-desingularization", which embeds fully faithfully (SOD) in all other desingularizations, and hence is unique.
- If $\pi : \widetilde{X} \to X$ is a crepant resolution, then $D^b(\widetilde{X})$ is the minimal desingularization of $D^b(X)$.

**Minimal Resolution Conjecture (Bondal–Orlov '02)**

- $D^b(X)$ has a minimal "nc-desingularization", which embeds fully faithfully (SOD) in all other desingularizations, and hence is unique.
- If $\pi : \widetilde{X} \to X$ is a crepant resolution, then $D^b(\widetilde{X})$ is the minimal desingularization of $D^b(X)$.

$\Longrightarrow$ "Categorical" MMP *always* stops at a smooth minimal model.
(crepant resolution = the categorical minimal model is geometric)

**Minimal Resolution Conjecture (Bondal–Orlov '02)**

- $D^b(X)$ has a minimal "nc-desingularization", which embeds fully faithfully (SOD) in all other desingularizations, and hence is unique.
- If $\pi : \widetilde{X} \to X$ is a crepant resolution, then $D^b(\widetilde{X})$ is the minimal desingularization of $D^b(X)$.

$\Longrightarrow$ "Categorical" MMP *always* stops at a smooth minimal model.
(crepant resolution = the categorical minimal model is geometric)

**Remark ("stacky crepant resolutions")**

By [BKR], crepant resolutions of certain (cDV) quotient singularities are $D$-equivalent to the (smooth) orbifold quotient $[X/G]$.

Generalizing this, $D^b([X/G])$ should always be the minimal desingularization for the coarse moduli space $X/G$.

## Warning: The minimal resolution conjecture is volatile

## Warning: The minimal resolution conjecture is volatile

1. Need to define "nc-desingularization".

**Warning: The minimal resolution conjecture is volatile**

1. Need to define "nc-desingularization".
2. It's not clear what the correct definition should be.

**Warning: The minimal resolution conjecture is volatile**

1. Need to define "nc-desingularization".
2. It's not clear what the correct definition should be.
3. With a naive definition, there *are* counter-examples. (arguably non-geometric ones)

**Warning: The minimal resolution conjecture is volatile**

1. Need to define "nc-desingularization".
2. It's not clear what the correct definition should be.
3. With a naive definition, there *are* counter-examples. (arguably non-geometric ones)

**Remark**

Even with a good definition, there is no clear proof approach; how do you construct a functor between arbitrary desingularizations?

# Other context

Why else do we care?

- Appears naturally in homological mirror symmetry (e.g. degenerations, or proving mirror symmetry for singularities via the resolution)
- Appears naturally in homological projective duality (e.g. K3 constructed from a singular cubic fourfold giving a categorical resolution of the Kuznetsov component).
- (Conjectural) unique minimal resolutions lead to (conjectural) invariants of singularities (like dual graphs of ADE surfaces).
- Like *DK*-conjecture, predicted equivalences between minimal resolutions can be *unexpected*, hinting at deeper structure in the constructions used to build the categories involved.
- There are actual hands-on constructions, allowing us to study concrete examples, and we find interesting structure which is poorly understood (e.g. null categories, relative singularity categories).

# Categorical resolutions

Suppose $\pi : \widetilde{X} \to X$ is a resolution of singularities.

# Categorical resolutions

Suppose $\pi : \widetilde{X} \to X$ is a resolution of singularities.

$$\begin{array}{ccc} D^b(\widetilde{X}) & & \\ {\scriptstyle \pi^*}\uparrow & \searrow^{\pi_*} & \\ \mathfrak{Perf}(X) & \hookrightarrow & D^b(X). \end{array}$$

# Categorical resolutions

Suppose $\pi : \widetilde{X} \to X$ is a resolution of singularities.

$$\begin{array}{ccc} D^b(\widetilde{X}) & & \\ {\scriptstyle \pi^*}\uparrow & \searrow^{\pi_*} & \\ \mathfrak{Perf}(X) & \hookrightarrow & D^b(X). \end{array}$$

If $\pi_*\pi^* = 1$ (i.e. $\pi^*$ fully faithful, $\pi_*$ ess. surjective) $\iff$ $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, we say $X$ has *rational singularities*.

# Categorical resolutions

Suppose $\pi : \widetilde{X} \to X$ is a resolution of singularities.

$$\begin{array}{ccc} D^b(\widetilde{X}) & & \\ {\scriptstyle \pi^*}\uparrow & \searrow^{\pi_*} & \\ \mathfrak{Perf}(X) & \hookrightarrow & D^b(X). \end{array}$$

If $\pi_*\pi^* = 1$ (i.e. $\pi^*$ fully faithful, $\pi_*$ ess. surjective) $\iff$ $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, we say $X$ has *rational singularities*.

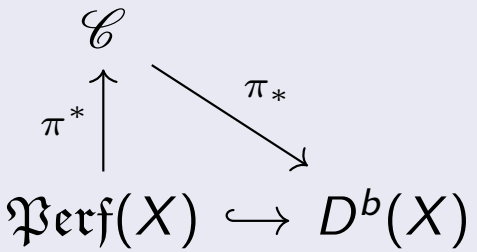
**Remark**

- Crepant resolution $\implies$ rational singularities.
- Terminal singularities $\implies$ rational singularities.

# Categorical resolutions

## Definition (Kuznetsov '08)

A *weak categorical resolution* is a smooth category $\mathscr{C}$ with adjoint functors




satisfying $\pi_*\pi^* = 1$.

# Categorical resolutions

## Definition (Kuznetsov '08)

A *weak categorical resolution* is a smooth category $\mathscr{C}$ with adjoint functors

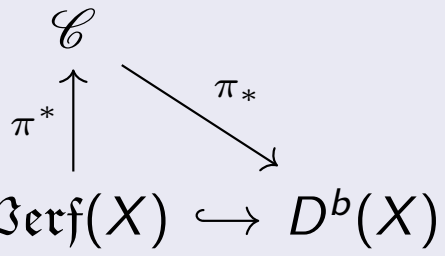
$$\begin{array}{ccc} & & \mathscr{C} \\ & {\scriptstyle \pi^*}\uparrow & \searrow^{\pi_*} \\ \mathfrak{Perf}(X) & \hookrightarrow & D^b(X) \end{array}$$

satisfying $\pi_*\pi^* = 1$. It is *weakly crepant* if $\pi^! = \pi^*$, where $\pi^* \dashv \pi_* \dashv \pi^!$.

# Categorical resolutions

## Definition (Kuznetsov '08)

A *weak categorical resolution* is a smooth category $\mathscr{C}$ with adjoint functors




satisfying $\pi_*\pi^* = 1$. It is *weakly crepant* if $\pi^! = \pi^*$, where $\pi^* \dashv \pi_* \dashv \pi^!$.

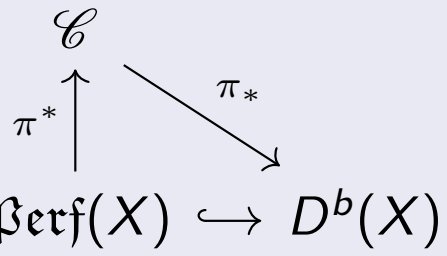
## Remark

- If $X$ has irrational singularities, $D^b(\widetilde{X})$ is *not* a categorical resolution!
- It is a weakly crepant categorical resolution iff $\pi : \widetilde{X} \to X$ is crepant.

# Categorical resolutions

## Definition (Kuznetsov '08)

A *weak categorical resolution* is a smooth category $\mathscr{C}$ with adjoint functors




satisfying $\pi_*\pi^* = 1$. It is *weakly crepant* if $\pi^! = \pi^*$, where $\pi^* \dashv \pi_* \dashv \pi^!$.

## Remark

- If $X$ has irrational singularities, $D^b(\widetilde{X})$ is *not* a categorical resolution!
- It is a weakly crepant categorical resolution iff $\pi : \widetilde{X} \to X$ is crepant.

The justification for this definition when $X$ has irrational singularities is a theorem of Kuznetsov and Lunts, which we will see in a moment.

# Categorical resolutions

Recall: $X$ is smooth iff $\mathfrak{Perf}(X)$ is smooth.

# Categorical resolutions

Recall: $X$ is smooth iff $\mathfrak{Perf}(X)$ is smooth.

**Theorem (Lunts '10)**

*The category $D^b(X)$ is smooth for any separated scheme $X$ of finite type over a perfect field, provided a dualizing complex exists.*

# Categorical resolutions

Recall: $X$ is smooth iff $\mathfrak{Perf}(X)$ is smooth.

**Theorem (Lunts '10)**

*The category $D^b(X)$ is smooth for any separated scheme $X$ of finite type over a perfect field, provided a dualizing complex exists.*

**Corollary**

*$\mathscr{C} = D^b(X)$, $\pi^* = \pi_* = 1$ is a universal weak categorical resolution.*

# Categorical resolutions

Recall: $X$ is smooth iff $\mathfrak{Perf}(X)$ is smooth.

**Theorem (Lunts '10)**

*The category $D^b(X)$ is smooth for any separated scheme $X$ of finite type over a perfect field, provided a dualizing complex exists.*

**Corollary**

$\mathscr{C} = D^b(X)$, $\pi^* = \pi_* = 1$ *is a universal weak categorical resolution.*

**Example**

Take $X = \operatorname{Spec} k[x]/x^2$. Then $D^b(X) = \langle k[x]/x \rangle$, and $\operatorname{Ext}^*_X(k,k) = k[\theta]$ where $|\theta| = 1$, so by Koszul duality $D^b(X) = D^b(k[\theta])$ which is smooth like $\mathbb{A}^1$.

# Categorical resolutions

Recall: $X$ is smooth iff $\mathfrak{Perf}(X)$ is smooth.

**Theorem (Lunts '10)**

*The category $D^b(X)$ is smooth for any separated scheme $X$ of finite type over a perfect field, provided a dualizing complex exists.*

**Corollary**

$\mathscr{C} = D^b(X)$, $\pi^* = \pi_* = 1$ *is a universal weak categorical resolution.*

**Example**

Take $X = \operatorname{Spec} k[x]/x^2$. Then $D^b(X) = \langle k[x]/x \rangle$, and $\operatorname{Ext}^*_X(k,k) = k[\theta]$ where $|\theta| = 1$, so by Koszul duality $D^b(X) = D^b(k[\theta])$ which is smooth like $\mathbb{A}^1$. This is not proper: $\dim_k(k[\theta]) = \infty$.

# Categorical resolutions

## Definition

A *proper categorical resolution* is a weak categorical resolution such that

- $\mathscr{C}$ is "linear over $D^b(X)$", i.e. $\mathrm{Hom}_{\mathscr{C}} = R\Gamma_X \circ \mathcal{H}om_{\mathscr{C}}$ where $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$, plus naturality conditions.
- The functors $\pi^*$, $\pi_*$ respect this structure.

Jargon: a module over the tensor category $(\mathfrak{Perf}(X), \otimes)$.

# Categorical resolutions

## Definition

A *proper categorical resolution* is a weak categorical resolution such that

- $\mathscr{C}$ is "linear over $D^b(X)$", i.e. $\mathrm{Hom}_{\mathscr{C}} = R\Gamma_X \circ \mathcal{H}om_{\mathscr{C}}$ where $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$, plus naturality conditions.
- The functors $\pi^*$, $\pi_*$ respect this structure.

Jargon: a module over the tensor category $(\mathfrak{Perf}(X), \otimes)$.

Idea: $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$ means Homs in $\mathscr{C}$ are finite rank (coherent) relative to $X \implies$ proper category relative to $X$.

# Categorical resolutions

## Definition

A *proper categorical resolution* is a weak categorical resolution such that

- $\mathscr{C}$ is "linear over $D^b(X)$", i.e. $\mathrm{Hom}_{\mathscr{C}} = R\Gamma_X \circ \mathcal{H}om_{\mathscr{C}}$ where $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$, plus naturality conditions.
- The functors $\pi^*$, $\pi_*$ respect this structure.

Jargon: a module over the tensor category $(\mathfrak{Perf}(X), \otimes)$.

Idea: $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$ means Homs in $\mathscr{C}$ are finite rank (coherent) relative to $X \implies$ proper category relative to $X$.

## Definition (Kuznetsov '08)

A *relative Serre functor* for a proper categorical resolution is an endofunctor $S : \mathscr{C} \to \mathscr{C}$ with a natural isomorphism

$$\mathcal{H}om_{\mathscr{C}}(x, S(y)) = \mathcal{H}om_X(\mathcal{H}om_{\mathscr{C}}(y, x), \mathcal{O}_X).$$

The resolution is *strongly crepant* if $\mathrm{id}_{\mathscr{C}}$ is a relative Serre functor.

# Categorical resolutions

**Remark**

It follows that $\pi^! = S \circ \pi^*$, so strongly crepant $\implies$ weakly crepant.

**Example**

For $\pi : \widetilde{X} \to X$ the relative Serre functor is $- \otimes \omega_{\widetilde{X}/X}[\dim \widetilde{X} - \dim X]$, and strong crepancy $\iff$ weak crepancy $\iff$ crepancy: $\omega_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}$.

**Theorem (Kuznetsov–Lunts '15)**

*Proper categorical resolutions exist for any separated scheme of finite type in characteristic zero.*

**Conjecture (Kuznetsov '08)**

Strongly crepant categorical resolutions are minimal.

# Example: dual numbers

Take $X = \operatorname{Spec} k[x]/x^2$ again. A proper categorical resolution has to be a smooth and proper category (since $k[x]/x^2$ is finite over $k$).

# Example: dual numbers

Take $X = \operatorname{Spec} k[x]/x^2$ again. A proper categorical resolution has to be a smooth and proper category (since $k[x]/x^2$ is finite over $k$).

Use the compactification $\mathbb{P}^1_{\theta:\phi}$, $|\theta| = 1$ of $D^b(X) = D^b(k[\theta])$. Beilinson:

$$\mathscr{C} = D^b\Big( \bullet \underset{\phi}{\overset{\theta}{\dashrightarrow}} \bullet \Big),$$

which contains $\mathfrak{Perf}(X)$ as the subcategory generated by $\mathcal{O}_{[0:1]} = \operatorname{cone}(\theta)$.

# Example: dual numbers

Take $X = \operatorname{Spec} k[x]/x^2$ again. A proper categorical resolution has to be a smooth and proper category (since $k[x]/x^2$ is finite over $k$).

Use the compactification $\mathbb{P}^1_{\theta:\phi}$, $|\theta| = 1$ of $D^b(X) = D^b(k[\theta])$. Beilinson:

$$\mathscr{C} = D^b\Big( \bullet \underset{\phi}{\overset{\theta}{\dashrightarrow}} \bullet \Big),$$

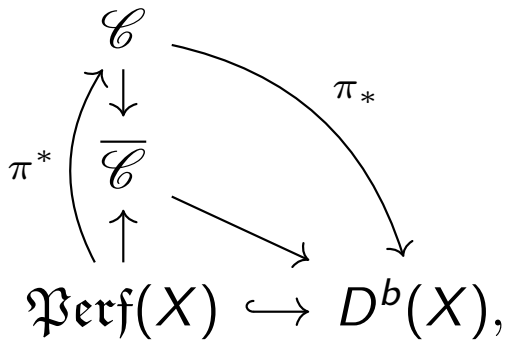
which contains $\mathfrak{Perf}(X)$ as the subcategory generated by $\mathcal{O}_{[0:1]} = \operatorname{cone}(\theta)$.

Two perspectives:

- Smoothing $\mathfrak{Perf}(X)$: finding a smooth category with an object having endomorphisms $k[x]/x^2$ (skyscraper sheaf in $\mathbb{P}^1$).
- Compactifying $D^b(X)$: compactifying $\mathbb{A}^1$ to $\mathbb{P}^1$.

# Minimality, SOD's and null categories

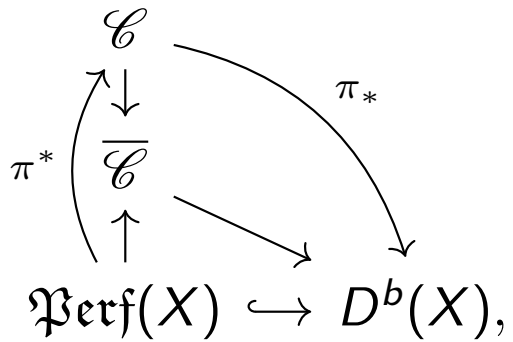
Suppose $\overline{\mathscr{C}}$ is the minimal resolution:




with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$.

# Minimality, SOD's and null categories

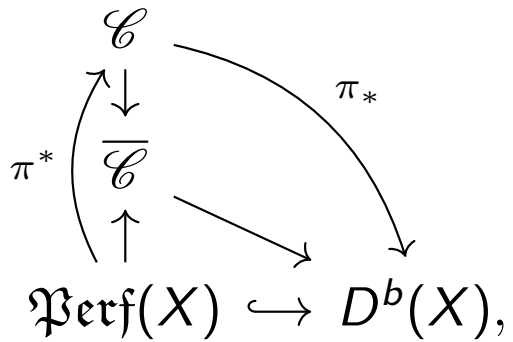
Suppose $\overline{\mathscr{C}}$ is the minimal resolution:




with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$. Then $\mathcal{A} \subseteq \ker \pi_*$

# Minimality, SOD's and null categories

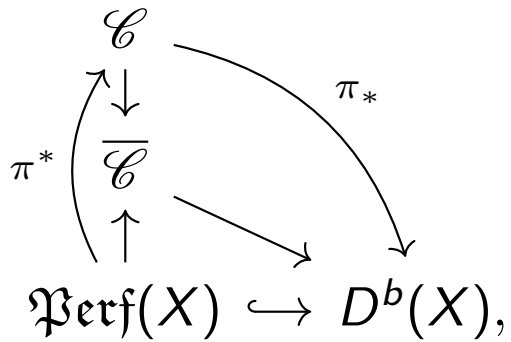
Suppose $\overline{\mathscr{C}}$ is the minimal resolution:




with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$. Then $\mathcal{A} \subseteq \ker \pi_* \implies \ker \pi_* = \langle \mathcal{K}, \mathcal{A} \rangle$ with $\mathcal{A}$ smooth and proper: "categorical absorption of singularities" for $\ker \pi_*$.

# Minimality, SOD's and null categories

Suppose $\overline{\mathscr{C}}$ is the minimal resolution:



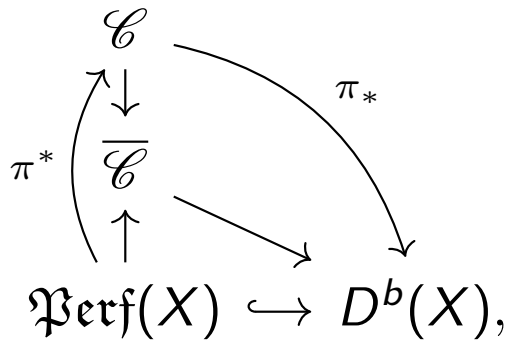

with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$. Then $\mathcal{A} \subseteq \ker \pi_* \implies \ker \pi_* = \langle \mathcal{K}, \mathcal{A} \rangle$ with $\mathcal{A}$ smooth and proper: "categorical absorption of singularities" for $\ker \pi_*$.

**Definition**

We call $\ker \pi_*$ the *null category* (or *kernel category*).

# Minimality, SOD's and null categories

Suppose $\overline{\mathscr{C}}$ is the minimal resolution:




with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$. Then $\mathcal{A} \subseteq \ker \pi_* \implies \ker \pi_* = \langle \mathcal{K}, \mathcal{A} \rangle$ with $\mathcal{A}$ smooth and proper: "categorical absorption of singularities" for $\ker \pi_*$.

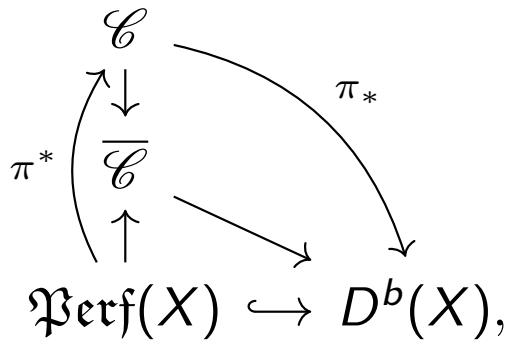
**Definition**

We call $\ker \pi_*$ the *null category* (or *kernel category*).

Note: $\mathscr{C}$ is minimal iff $\mathcal{A} = 0$ is the only possible absorption for $\ker \pi_*$.

# Minimality, SOD's and null categories

Suppose $\overline{\mathscr{C}}$ is the minimal resolution:




with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$. Then $\mathcal{A} \subseteq \ker \pi_* \implies \ker \pi_* = \langle \mathcal{K}, \mathcal{A} \rangle$ with $\mathcal{A}$ smooth and proper: "categorical absorption of singularities" for $\ker \pi_*$.

**Definition**

We call $\ker \pi_*$ the *null category* (or *kernel category*).

Note: $\mathscr{C}$ is minimal iff $\mathcal{A} = 0$ is the only possible absorption for $\ker \pi_*$. This is implied by the Calabi–Yau condition $S|_{\ker \pi_*} = [n]$, a weakening of strong crepancy, when $\ker \pi_*$ is connected.

# Minimality, SOD's and null categories

## Example

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

# Minimality, SOD's and null categories

## Example

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

# Minimality, SOD's and null categories

**Example**

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

Conversely, suppose we have a categorical absorption for $X$:
$D^b(X) = \langle \mathcal{X}, \mathcal{A} \rangle$ with $\mathcal{A} \subset \mathfrak{Perf}(X)$.

# Minimality, SOD's and null categories

**Example**

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

Conversely, suppose we have a categorical absorption for $X$: $D^b(X) = \langle \mathcal{X}, \mathcal{A} \rangle$ with $\mathcal{A} \subset \mathfrak{Perf}(X)$. Then $\mathcal{A} \subset \mathfrak{Perf}(X) \subset \mathscr{C}$, giving $\mathscr{C} = \langle \mathscr{C}', \mathcal{A} \rangle$, where $\mathscr{C}'$ is a resolution for $\mathcal{X}$.

# Minimality, SOD's and null categories

## Example

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

Conversely, suppose we have a categorical absorption for $X$: $D^b(X) = \langle \mathcal{X}, \mathcal{A} \rangle$ with $\mathcal{A} \subset \mathfrak{Perf}(X)$. Then $\mathcal{A} \subset \mathfrak{Perf}(X) \subset \mathscr{C}$, giving $\mathscr{C} = \langle \mathscr{C}', \mathcal{A} \rangle$, where $\mathscr{C}'$ is a resolution for $\mathcal{X}$. Conversely, any such $\mathscr{C}'$ can be glued to $\mathcal{A}$ to give a resolution for $D^b(X)$ with the same null category.

# Minimality, SOD's and null categories

**Example**

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

Conversely, suppose we have a categorical absorption for $X$: $D^b(X) = \langle \mathcal{X}, \mathcal{A} \rangle$ with $\mathcal{A} \subset \mathfrak{Perf}(X)$. Then $\mathcal{A} \subset \mathfrak{Perf}(X) \subset \mathscr{C}$, giving $\mathscr{C} = \langle \mathscr{C}', \mathcal{A} \rangle$, where $\mathscr{C}'$ is a resolution for $\mathcal{X}$. Conversely, any such $\mathscr{C}'$ can be glued to $\mathcal{A}$ to give a resolution for $D^b(X)$ with the same null category.

$$\{\text{categorical resolutions of } D^b(X)\} = \{\text{categorical resolutions of the absorbing subcategory } \mathcal{X}\}.$$

# Minimality, SOD's and null categories

**Example**

For the resolution $\mathbb{P}^1_{\theta:\phi}$ of the dual numbers, the null category is generated by cone($\phi$), with endomorphisms $k[y]/y^2$ where $|y| = 2$.

Nodal singularities in all dimensions have categorical resolutions with null categories generated by a (2 or 3)-spherical object. (Cattani et. al. '23)

Conversely, suppose we have a categorical absorption for $X$: $D^b(X) = \langle \mathcal{X}, \mathcal{A} \rangle$ with $\mathcal{A} \subset \mathfrak{Perf}(X)$. Then $\mathcal{A} \subset \mathfrak{Perf}(X) \subset \mathscr{C}$, giving $\mathscr{C} = \langle \mathscr{C}', \mathcal{A} \rangle$, where $\mathscr{C}'$ is a resolution for $\mathcal{X}$. Conversely, any such $\mathscr{C}'$ can be glued to $\mathcal{A}$ to give a resolution for $D^b(X)$ with the same null category.

$$\{\text{categorical resolutions of } D^b(X)\} = \{\text{categorical resolutions of the absorbing subcategory } \mathcal{X}\}.$$

Note: if $\mathcal{X} = D^b(R)$, we get $D_{\mathrm{Sg}}(X) = D_{\mathrm{Sg}}(R)$.

# Constructions

## Auslander algebras (Kuznetsov–Lunts)

If $S$ is a non-reduced thickening of a smooth variety $S_0$, there is a sheaf of algebras $\mathcal{A}_{S/S_0}$ on $S$ which is a proper categorical resolution, with an SOD:

$$D^b(\mathcal{A}_{S/S_0}) = \langle D^b(S_0), \ldots, D^b(S_0) \rangle.$$

The number of components corresponds to the order of thickening.

# Constructions

## Auslander algebras (Kuznetsov–Lunts)

If $S$ is a non-reduced thickening of a smooth variety $S_0$, there is a sheaf of algebras $\mathcal{A}_{S/S_0}$ on $S$ which is a proper categorical resolution, with an SOD:

$$D^b(\mathcal{A}_{S/S_0}) = \langle D^b(S_0), \dots, D^b(S_0) \rangle.$$

The number of components corresponds to the order of thickening.

## Example

The dual numbers $k[x]/x^2$ are a 2nd order thickening of a point, so we get an SOD: $D^b(\mathcal{A}) = \langle D^b(\mathrm{pt}), D^b(\mathrm{pt}) \rangle$. The Auslander resolution $D^b(\mathcal{A})$ is equivalent to the $\mathbb{P}^1_{\theta:\phi}$ resolution.

# Constructions

## Auslander algebras (Kuznetsov–Lunts)

If $S$ is a non-reduced thickening of a smooth variety $S_0$, there is a sheaf of algebras $\mathcal{A}_{S/S_0}$ on $S$ which is a proper categorical resolution, with an SOD:

$$D^b(\mathcal{A}_{S/S_0}) = \langle D^b(S_0), \dots, D^b(S_0) \rangle.$$

The number of components corresponds to the order of thickening.

## Example

The dual numbers $k[x]/x^2$ are a 2nd order thickening of a point, so we get an SOD: $D^b(\mathcal{A}) = \langle D^b(\mathrm{pt}), D^b(\mathrm{pt}) \rangle$. The Auslander resolution $D^b(\mathcal{A})$ is equivalent to the $\mathbb{P}^1_{\theta:\phi}$ resolution.

The Kuznetsov–Lunts existence theorem is proved by gluing Auslander resolutions of thickenings of the blowup centres appearing in a resolution of singularities.

# Constructions

## NC(C)R's (Van den Bergh)

In nice situations (e.g. quotient singularities), when $X = \operatorname{Spec} R$, there can be a reflexive $R$-module $M$ such that $\Lambda = \operatorname{End}_R(M)$ has good homological properties.

# Constructions

## NC(C)R's (Van den Bergh)

In nice situations (e.g. quotient singularities), when $X = \operatorname{Spec} R$, there can be a reflexive $R$-module $M$ such that $\Lambda = \operatorname{End}_R(M)$ has good homological properties. It often gives a strongly crepant categorical resolution $D^b(\Lambda)$.

# Constructions

## NC(C)R's (Van den Bergh)

In nice situations (e.g. quotient singularities), when $X = \operatorname{Spec} R$, there can be a reflexive $R$-module $M$ such that $\Lambda = \operatorname{End}_R(M)$ has good homological properties. It often gives a strongly crepant categorical resolution $D^b(\Lambda)$.

These "non-commutative resolutions" are a representation-theoretic construction which can be viewed as giving special categorical resolutions which have tilting objects.

# Constructions

## NC(C)R's (Van den Bergh)

In nice situations (e.g. quotient singularities), when $X = \operatorname{Spec} R$, there can be a reflexive $R$-module $M$ such that $\Lambda = \operatorname{End}_R(M)$ has good homological properties. It often gives a strongly crepant categorical resolution $D^b(\Lambda)$.

These "non-commutative resolutions" are a representation-theoretic construction which can be viewed as giving special categorical resolutions which have tilting objects.

## Caveat

NCCR's are not known to always give categorical resolutions due to a technical obstruction, but it is expected and known in many cases.

# Matrix factorizations

## Definition

A *Landau–Ginzburg model* is a smooth variety $U$ with a $\mathbb{C}^*$-action (*R-charge*) and a global function $W \in H^0(U, \mathcal{O}_U)$ (the *superpotential*) with is $\mathbb{C}^*$-equivariant of weight 2.

# Matrix factorizations

## Definition

A *Landau–Ginzburg model* is a smooth variety $U$ with a $\mathbb{C}^*$-action (*R-charge*) and a global function $W \in H^0(U, \mathcal{O}_U)$ (the *superpotential*) with is $\mathbb{C}^*$-equivariant of weight 2. Equivalently, the coordinate ring $\mathcal{O}_U$ is graded such that $|W| = 2$.

# Matrix factorizations

## Definition

A *Landau–Ginzburg model* is a smooth variety $U$ with a $\mathbb{C}^*$-action (*R-charge*) and a global function $W \in H^0(U, \mathcal{O}_U)$ (the *superpotential*) with is $\mathbb{C}^*$-equivariant of weight 2. Equivalently, the coordinate ring $\mathcal{O}_U$ is graded such that $|W| = 2$.

## Definition

The (derived) category of *matrix factorizations* on a Landau–Ginzburg model is denoted $\mathrm{MF}(U, W)$, with objects given by $\mathbb{C}^*$-equivariant sheaves with $\mathbb{C}^*$-equivariant endomorphisms $d$ satisfying $d^2 = W \cdot \mathrm{id}$.

# Matrix factorizations

## Definition

A *Landau–Ginzburg model* is a smooth variety $U$ with a $\mathbb{C}^*$-action (*R-charge*) and a global function $W \in H^0(U, \mathcal{O}_U)$ (the *superpotential*) with is $\mathbb{C}^*$-equivariant of weight 2. Equivalently, the coordinate ring $\mathcal{O}_U$ is graded such that $|W| = 2$.

## Definition

The (derived) category of *matrix factorizations* on a Landau–Ginzburg model is denoted $\mathrm{MF}(U, W)$, with objects given by $\mathbb{C}^*$-equivariant sheaves with $\mathbb{C}^*$-equivariant endomorphisms $d$ satisfying $d^2 = W \cdot \mathrm{id}$.

## Fact

Because $W = \frac{1}{2}dd + \frac{1}{2}dd$ and $\partial_x(W) = \partial_x(d)d + d\partial_x(d)$ are null-homotopies, the category $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W) \subset U$.

# Matrix factorizations

## Facts

- $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W)$.

# Matrix factorizations

## Facts

- $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W)$.
- $\mathrm{MF}(U, W)$ is smooth, since we assumed $U$ is smooth.

# Matrix factorizations

## Facts

- $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W)$.
- $\mathrm{MF}(U, W)$ is smooth, since we assumed $U$ is smooth.
- $\mathrm{MF}(U, W)$ is proper iff $\mathrm{Crit}(W)$ is proper.

# Matrix factorizations

## Facts

- $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W)$.
- $\mathrm{MF}(U, W)$ is smooth, since we assumed $U$ is smooth.
- $\mathrm{MF}(U, W)$ is proper iff $\mathrm{Crit}(W)$ is proper.
- The Serre functor on $\mathrm{MF}(U, W)$ is a shift of $- \otimes \omega_U$.

# Matrix factorizations

## Facts

- $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W)$.
- $\mathrm{MF}(U, W)$ is smooth, since we assumed $U$ is smooth.
- $\mathrm{MF}(U, W)$ is proper iff $\mathrm{Crit}(W)$ is proper.
- The Serre functor on $\mathrm{MF}(U, W)$ is a shift of $- \otimes \omega_U$.

## Example (Classical Knörrer periodicity)

$\mathcal{E} = \left( \mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}[-1] \right) \in \mathrm{MF}(\mathbb{A}^2, xy)$ is an exceptional object, where $R$-charge is $|x| = 0$, $|y| = 2$. It is quasi-isomorphic to $\mathcal{O}/y$, so

$$\mathcal{H}om(\mathcal{E}, \mathcal{E}) = \mathcal{H}om\left( \mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}[-1] \ , \mathcal{O}/y \right)$$

$$= (\mathcal{O}/y[1] \xrightarrow{x} \mathcal{O}/y) = \mathcal{O}/(x, y).$$

This is a sheaf on $\mathrm{Crit}(xy) = \{x = y = 0\}$, as expected.

# Matrix factorizations

**Theorem (Knörrer periodicity (Orlov '06, ...))**

*Suppose $X \subset U$ is a hypersurface cut out by $f \in H^0(U, \mathcal{L})$. Then*

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}\, \mathcal{L}^\vee, fp); \qquad \mathscr{F} \mapsto \Big( \mathscr{F} \underset{p}{\overset{f}{\rightleftarrows}} \mathscr{F} \otimes \mathcal{L}[-1] \Big),$$

*where the fiber coordinate $p \in H^0(\mathrm{Tot}\, \mathcal{L}^\vee, \mathcal{L}^\vee)$ is given R-charge $|p| = 2$.*

**Remark**

Here $\mathrm{Crit}(fp) = \{f = p = 0\} \cup \{\mathrm{Crit}(f)\} = X \cup \mathcal{L}^\vee|_{\mathrm{Sing}(X)}$.

# Matrix factorizations

**Theorem (Knörrer periodicity (Orlov '06, ...))**

*Suppose $X \subset U$ is a hypersurface cut out by $f \in H^0(U, \mathcal{L})$. Then*

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}\, \mathcal{L}^\vee, fp); \qquad \mathscr{F} \mapsto \Big( \mathscr{F} \underset{p}{\overset{f}{\rightleftarrows}} \mathscr{F} \otimes \mathcal{L}[-1] \Big),$$

*where the fiber coordinate $p \in H^0(\mathrm{Tot}\, \mathcal{L}^\vee, \mathcal{L}^\vee)$ is given R-charge $|p| = 2$.*

**Remark**

Here $\mathrm{Crit}(fp) = \{f = p = 0\} \cup \{\mathrm{Crit}(f)\} = X \cup \mathcal{L}^\vee|_{\mathrm{Sing}(X)}$.
This is proper over $X$ iff $X$ is smooth.

# Matrix factorizations

**Theorem (Knörrer periodicity (Orlov '06, ...))**

*Suppose $X \subset U$ is a hypersurface cut out by $f \in H^0(U, \mathcal{L})$. Then*

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}\, \mathcal{L}^\vee, fp); \qquad \mathscr{F} \mapsto \Big( \mathscr{F} \underset{p}{\overset{f}{\rightleftarrows}} \mathscr{F} \otimes \mathcal{L}[-1] \Big),$$

*where the fiber coordinate $p \in H^0(\mathrm{Tot}\, \mathcal{L}^\vee, \mathcal{L}^\vee)$ is given R-charge $|p| = 2$.*

**Remark**

Here $\mathrm{Crit}(fp) = \{f = p = 0\} \cup \{\mathrm{Crit}(f)\} = X \cup \mathcal{L}^\vee|_{\mathrm{Sing}(X)}$.
This is proper over $X$ iff $X$ is smooth.
$\implies$ matches the weak categorical resolution $D^b(X)$.

# Matrix factorizations

**Theorem (Knörrer periodicity (Orlov '06, ...))**

*Suppose $X \subset U$ is a hypersurface cut out by $f \in H^0(U, \mathcal{L})$. Then*

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}\,\mathcal{L}^\vee, fp); \qquad \mathscr{F} \mapsto \Big( \mathscr{F} \underset{p}{\overset{f}{\rightleftarrows}} \mathscr{F} \otimes \mathcal{L}[-1] \Big),$$

*where the fiber coordinate $p \in H^0(\mathrm{Tot}\,\mathcal{L}^\vee, \mathcal{L}^\vee)$ is given R-charge $|p| = 2$.*

**Remark**

Here $\mathrm{Crit}(fp) = \{f = p = 0\} \cup \{\mathrm{Crit}(f)\} = X \cup \mathcal{L}^\vee|_{\mathrm{Sing}(X)}$.
This is proper over $X$ iff $X$ is smooth.
$\implies$ matches the weak categorical resolution $D^b(X)$.

**Idea ("Exoflops" Aspinwall '14)**

We find proper categorical resolutions of $X$ by compactifying $\mathrm{Crit}(fp)$ inside $\mathrm{Tot}\,\mathcal{L}^\vee$, i.e. partially compactifying $\mathrm{Tot}\,\mathcal{L}^\vee$.

## Remark

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

$$\left( \mathcal{E} \underset{p}{\overset{f}{\rightleftarrows}} \mathcal{E} \otimes \mathcal{L}[-1] \right) = \mathcal{E}/p,$$

and in fact $\mathfrak{Perf}(X) = \mathrm{MF}_{\{p=0\}}(\mathrm{Tot}\,\mathcal{L}^\vee, fp)$ is the subcategory supported at $\{p = 0\}$.

## Remark

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

$$\left( \mathcal{E} \underset{p}{\overset{f}{\rightleftarrows}} \mathcal{E} \otimes \mathcal{L}[-1] \right) = \mathcal{E}/p,$$

and in fact $\mathfrak{Perf}(X) = \mathrm{MF}_{\{p=0\}}(\mathrm{Tot}\,\mathcal{L}^\vee, fp)$ is the subcategory supported at $\{p = 0\}$. The "geometric" part of the critical locus $\mathrm{Crit}(fp)$.

## Remark

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

$$\left( \mathcal{E} \underset{p}{\overset{f}{\rightleftarrows}} \mathcal{E} \otimes \mathcal{L}[-1] \right) = \mathcal{E}/p,$$

and in fact $\mathfrak{Perf}(X) = \mathrm{MF}_{\{p=0\}}(\mathrm{Tot}\,\mathcal{L}^\vee, fp)$ is the subcategory supported at $\{p = 0\}$. The "geometric" part of the critical locus $\mathrm{Crit}(fp)$.

## Remark

If $j : U \hookrightarrow \overline{U}$ is a partial compactification, then $j_* \dashv j^* \dashv j_*$.

**Remark**

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

$$\left( \mathcal{E} \underset{p}{\overset{f}{\rightleftarrows}} \mathcal{E} \otimes \mathcal{L}[-1] \right) = \mathcal{E}/p,$$

and in fact $\mathfrak{Perf}(X) = \mathrm{MF}_{\{p=0\}}(\mathrm{Tot}\,\mathcal{L}^\vee, fp)$ is the subcategory supported at $\{p = 0\}$. The "geometric" part of the critical locus $\mathrm{Crit}(fp)$.

**Remark**

If $j : U \hookrightarrow \overline{U}$ is a partial compactification, then $j_* \dashv j^* \dashv j_*$. $\implies$ this method produces weakly crepant proper categorical resolutions.

**Remark**

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

$$\left( \mathcal{E} \underset{p}{\overset{f}{\rightleftarrows}} \mathcal{E} \otimes \mathcal{L}[-1] \right) = \mathcal{E}/p,$$

and in fact $\mathfrak{Perf}(X) = \mathrm{MF}_{\{p=0\}}(\mathrm{Tot}\,\mathcal{L}^\vee, fp)$ is the subcategory supported at $\{p = 0\}$. The "geometric" part of the critical locus $\mathrm{Crit}(fp)$.

**Remark**

If $j : U \hookrightarrow \overline{U}$ is a partial compactification, then $j_* \dashv j^* \dashv j_*$. $\implies$ this method produces weakly crepant proper categorical resolutions.

**Remark**

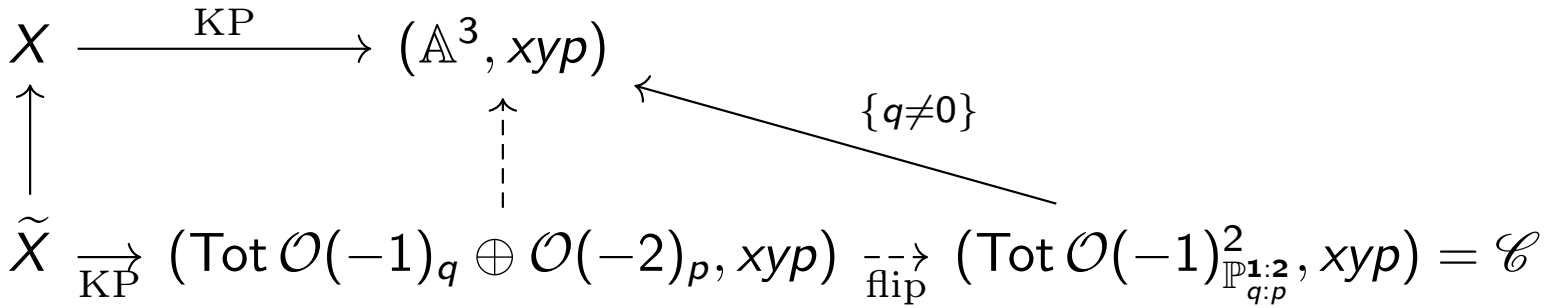
By Knörrer periodicity $f \rightsquigarrow f + x^2 + y^2$, studying curves / surfaces also gives results for higher-dimensional hypersurfaces.

# Example $A_1$

Node $X = \{xy = 0\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2 = \mathrm{Tot}\, \mathcal{O}(-1)_{\mathbb{P}^1}$.

# Example $A_1$

Node $X = \{xy = 0\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2 = \mathrm{Tot}\, \mathcal{O}(-1)_{\mathbb{P}^1}$.

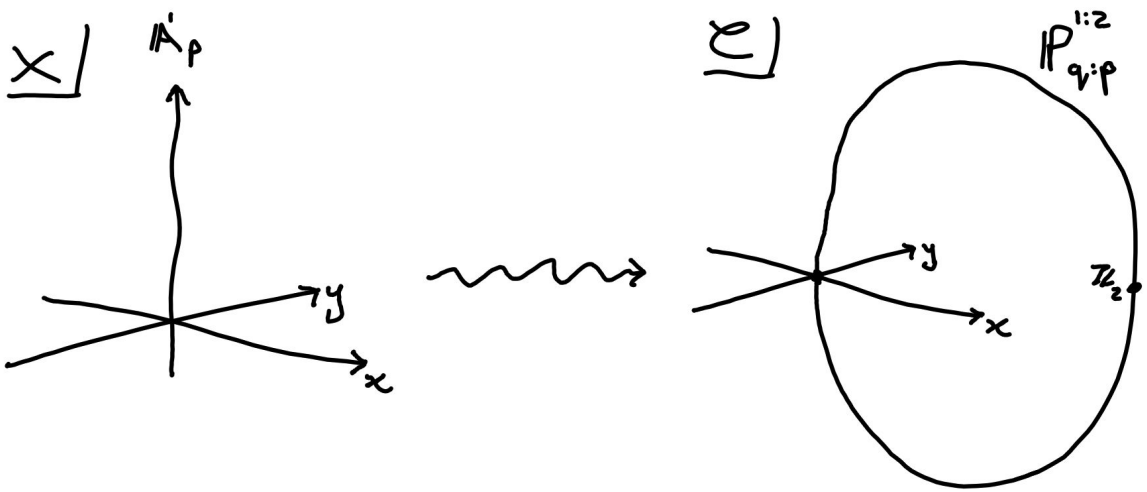


# Example $A_1$

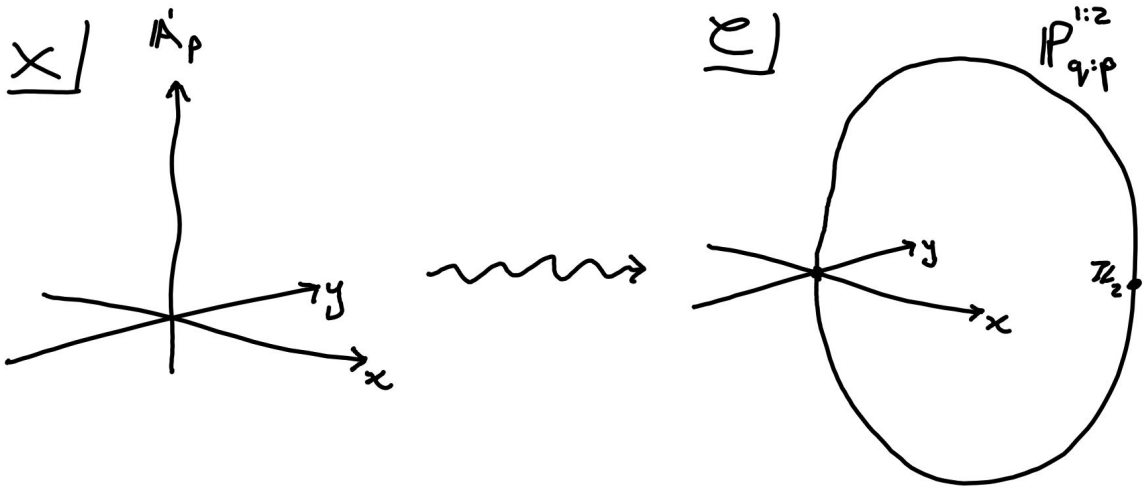
Node $X = \{xy = 0\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2 = \mathrm{Tot}\, \mathcal{O}(-1)_{\mathbb{P}^1}$.

$$\begin{array}{ccccc} X & \xrightarrow{\quad\mathrm{KP}\quad} & (\mathbb{A}^3, xyp) & \xleftarrow{\quad\{q\neq 0\}\quad} & \\ \uparrow & & \uparrow & & \\ \widetilde{X} & \xrightarrow[\mathrm{KP}]{} & (\mathrm{Tot}\, \mathcal{O}(-1)_q \oplus \mathcal{O}(-2)_p, xyp) & \xrightarrow[\mathrm{flip}]{} & (\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}_{q:p}}, xyp) = \mathscr{C} \end{array}$$

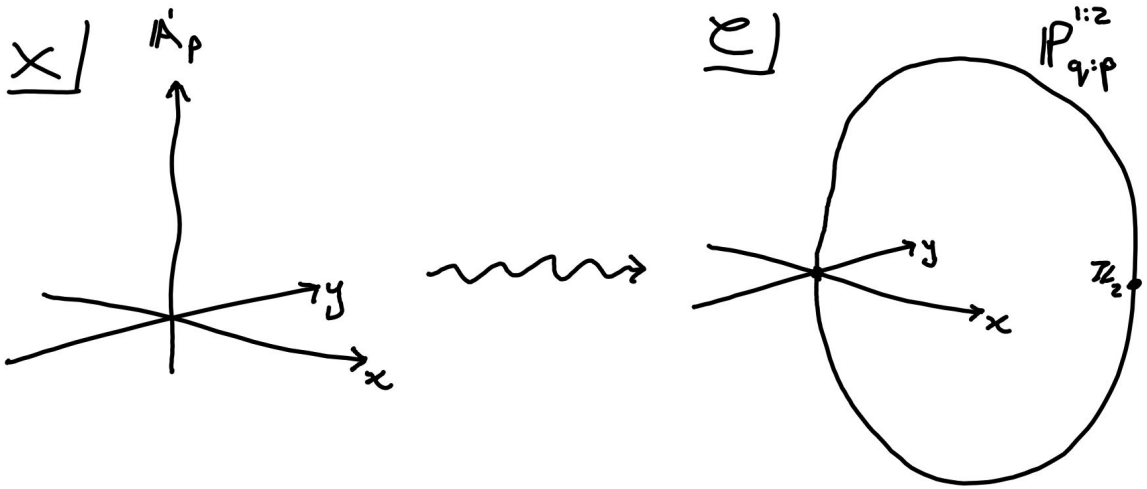
Here $\mathscr{C} = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle$ matches Kuznetsov–Lunts gluing.



# Example $A_1$

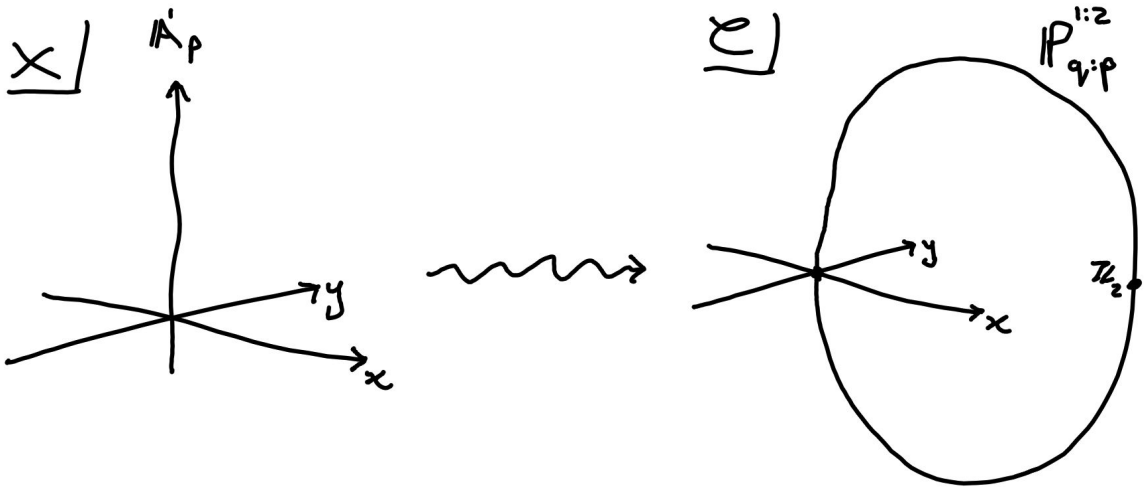


# Example $A_1$




- At $p = 0$: $\mathfrak{Perf}(X)$
- At $q = 0$: $\ker \pi_* = D^b_0([\mathbb{A}^1/\mathbb{Z}_2])$.

# Example $A_1$

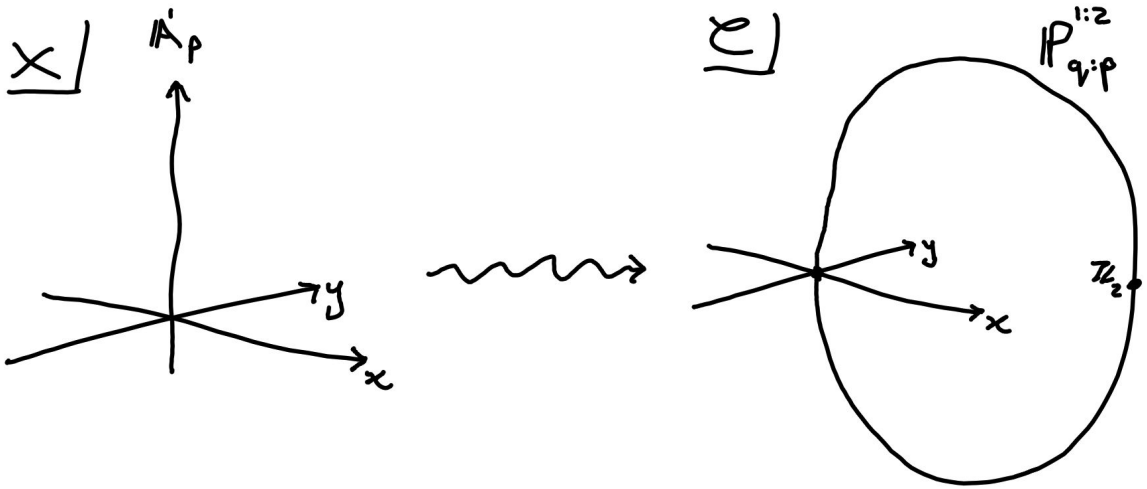



- At $p = 0$: $\mathfrak{Perf}(X)$
- At $q = 0$: $\ker \pi_* = D^b_0([\mathbb{A}^1/\mathbb{Z}_2])$.

**Fact**

$D^b([\mathbb{A}^1/\mathbb{Z}_2]) = \langle \mathcal{O}, \mathcal{O}_0 \rangle = \langle D^b(\mathbb{A}^1), D^b(\mathrm{pt}) \rangle$

# Example $A_1$




- At $p = 0$: $\mathfrak{Perf}(X)$
- At $q = 0$: $\ker \pi_* = D_0^b([\mathbb{A}^1/\mathbb{Z}_2])$.

**Fact**

$D^b([\mathbb{A}^1/\mathbb{Z}_2]) = \langle \mathcal{O}, \mathcal{O}_0 \rangle = \langle D^b(\mathbb{A}^1), D^b(\mathrm{pt}) \rangle$
$\implies D_0^b([\mathbb{A}^1/\mathbb{Z}_2]) = \langle D_0^b(\mathbb{A}^1), D^b(\mathrm{pt}) \rangle$ absorption!

# Example $A_1$

So $\mathscr{C} = \langle \overline{\mathscr{C}}, D^b(\mathrm{pt}) \rangle$.

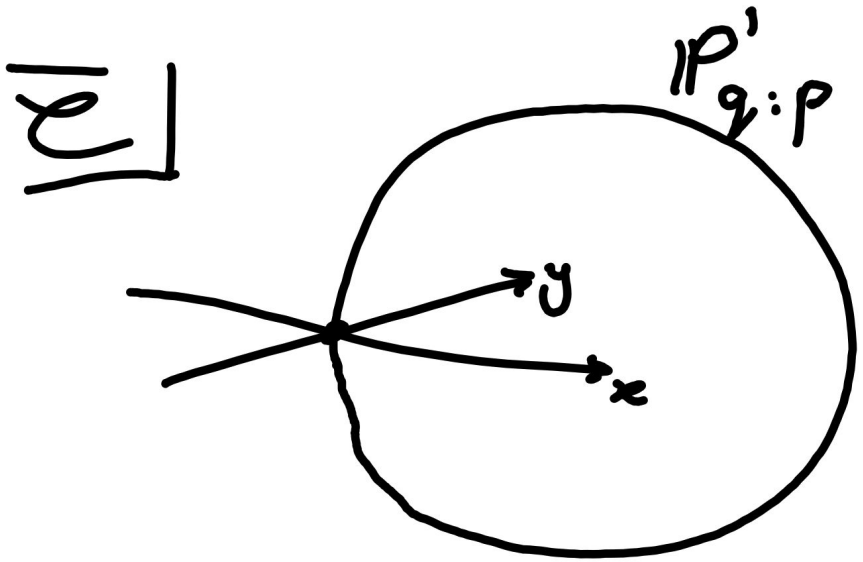
# Example $A_1$

So $\mathscr{C} = \langle \overline{\mathscr{C}}, D^b(\mathrm{pt}) \rangle$. We can get $\overline{\mathscr{C}}$ directly:

# Example $A_1$

So $\mathscr{C} = \langle \overline{\mathscr{C}}, D^b(\mathrm{pt}) \rangle$. We can get $\overline{\mathscr{C}}$ directly:

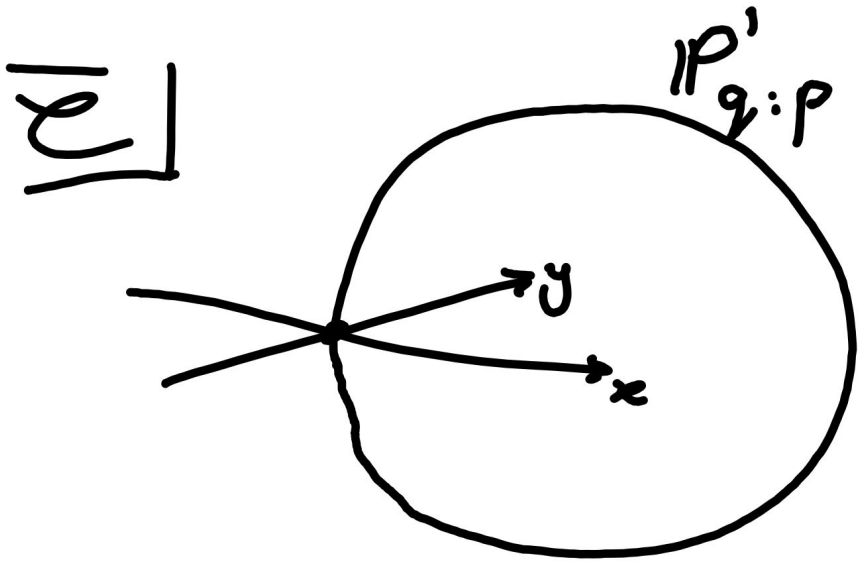
$$\overline{\mathscr{C}} = \mathrm{MF}(\mathrm{Tot}(\mathcal{O}_x \oplus \mathcal{O}(-1)_y \to \mathbb{P}^1_{q:p}), xyp) \xrightarrow{\{q \neq 0\}} \mathrm{MF}(\mathbb{A}^3, xyp)$$



# Example $A_1$

So $\mathscr{C} = \langle \overline{\mathscr{C}}, D^b(\mathrm{pt}) \rangle$. We can get $\overline{\mathscr{C}}$ directly:

$$\overline{\mathscr{C}} = \mathrm{MF}(\mathrm{Tot}(\mathcal{O}_x \oplus \mathcal{O}(-1)_y \to \mathbb{P}^1_{q:p}), xyp) \xrightarrow{\{q \neq 0\}} \mathrm{MF}(\mathbb{A}^3, xyp)$$




The null category at $\{q = 0\}$ is $D^b_0(\mathbb{A}^1) = \mathfrak{Perf}(k[\theta]/\theta^2)$, via Koszul duality (work out $R$-charge $\implies |\theta| = 3 \implies$ 3-spherical object).

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p \neq 0\}}(\mathrm{Tot}\, \mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p \neq 0\}}(\mathrm{Tot}\, \mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p\neq 0\}}(\mathrm{Tot}\,\mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p\neq 0\}}(\mathrm{Tot}\,\mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole. The choice of resolution $\mathscr{C}$ compactifies the $p = \infty$ hole.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p\neq 0\}}(\mathrm{Tot}\,\mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole. The choice of resolution $\mathscr{C}$ compactifies the $p = \infty$ hole. These are geometrically independent!

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p \neq 0\}}(\mathrm{Tot}\, \mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole. The choice of resolution $\mathscr{C}$ compactifies the $p = \infty$ hole. These are geometrically independent!

## Conjecture

- Categorical resolutions of isolated hypersurface singularities should be linear over $\mathbb{P}^1$ in this way.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p\neq 0\}}(\mathrm{Tot}\,\mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole. The choice of resolution $\mathscr{C}$ compactifies the $p = \infty$ hole. These are geometrically independent!

## Conjecture

- Categorical resolutions of isolated hypersurface singularities should be linear over $\mathbb{P}^1$ in this way. The $\{p \neq 0\}$ part is $\mathscr{C}/\mathfrak{Perf}(X)$, the $\{q \neq 0\}$ part is $D^b(X)$, and they intersect in $D_{\mathrm{Sg}}(X)$.

## Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathrm{MF}_{\{p\neq 0\}}(\mathrm{Tot}\,\mathcal{L}^{\vee}, fp)$ is the branch bit we are seeing, separated from $X$, pre-compactification. For the node this is $\mathbb{A}^1_p - \{0\}$, i.e. $\mathbb{C}[p^{\pm 1}]$, with $|p| = 2$.

The choice of $X$ with this particular $D_{\mathrm{Sg}}$ compactifies the $p = 0$ hole. The choice of resolution $\mathscr{C}$ compactifies the $p = \infty$ hole. These are geometrically independent!

## Conjecture

- Categorical resolutions of isolated hypersurface singularities should be linear over $\mathbb{P}^1$ in this way. The $\{p \neq 0\}$ part is $\mathscr{C}/\mathfrak{Perf}(X)$, the $\{q \neq 0\}$ part is $D^b(X)$, and they intersect in $D_{\mathrm{Sg}}(X)$.
- The data of $\mathscr{C}/\mathfrak{Perf}(X) \to D_{\mathrm{Sg}}(X)$ should then determine the resolution via gluing with $D^b(X) \to D_{\mathrm{Sg}}(X)$.
- In particular, if $D_{\mathrm{Sg}}(X') = D_{\mathrm{Sg}}(X)$ then $X'$ and $X$ should have the same categorical resolution theory.

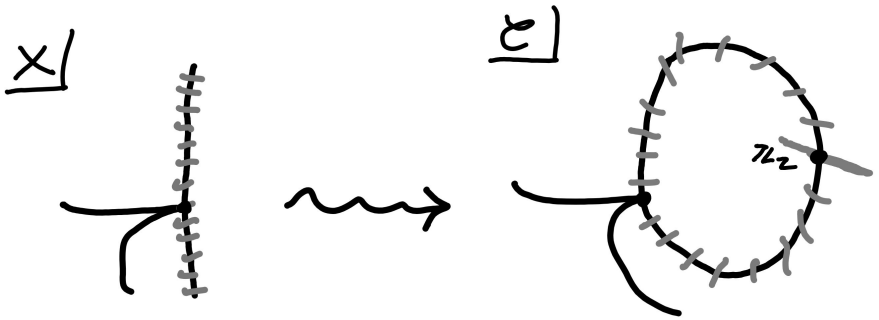
# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$.

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$

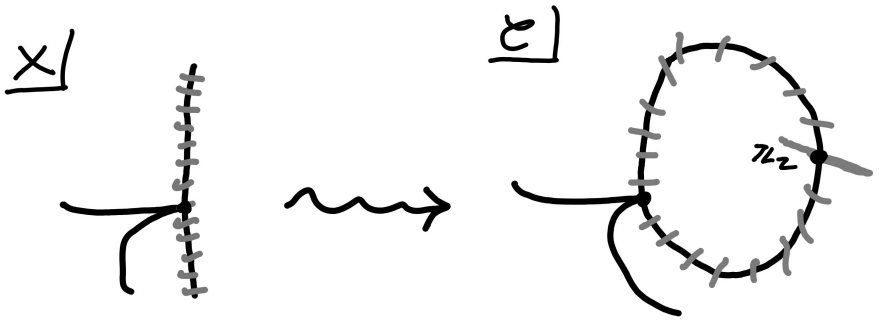



Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$



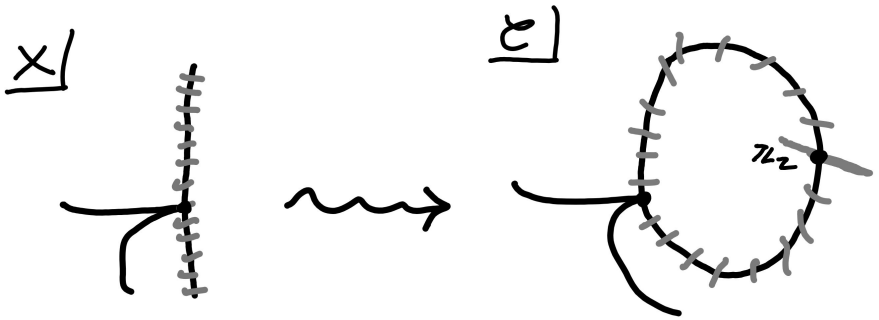

Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q)$$

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$



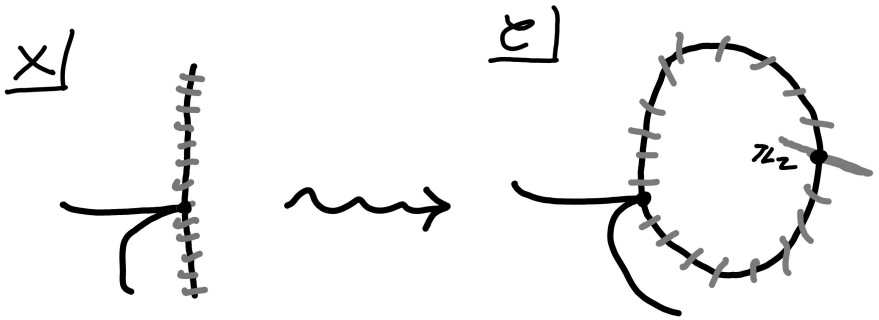

Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q) = \mathrm{MF}(\mathbb{A}^2, x^3 q)$$

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$



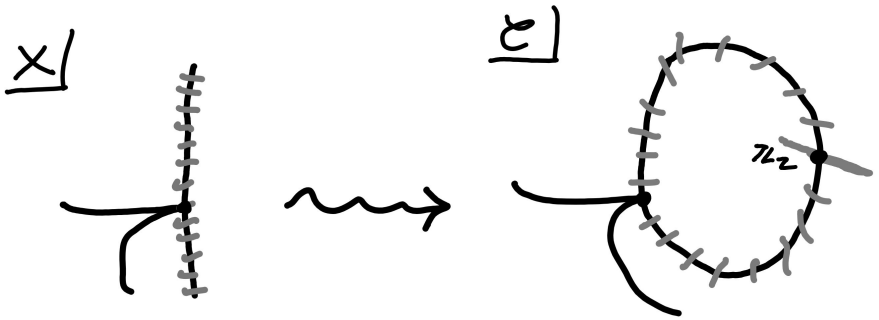

Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q) = \mathrm{MF}(\mathbb{A}^2, x^3 q) = D^b(k[x]/x^3),$$

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$




Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

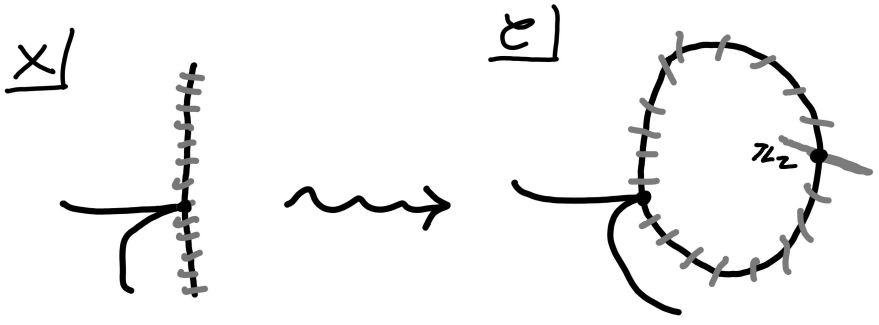
$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q) = \mathrm{MF}(\mathbb{A}^2, x^3 q) = D^b(k[x]/x^3),$$

and the null category at $\{q = 0\}$ is $\mathfrak{Perf}(\mathbb{C}[x]/x^3)$;

# Example $A_2$

Cusp $X = \{y^2 = x^3\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0 \mathbb{A}^2$. We get

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}\, \mathcal{O}(-1)^2_{\mathbb{P}^{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt}) \rangle.$$




Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

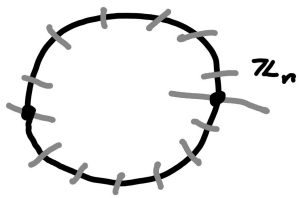
$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q) = \mathrm{MF}(\mathbb{A}^2, x^3 q) = D^b(k[x]/x^3),$$

and the null category at $\{q = 0\}$ is $\mathfrak{Perf}(\mathbb{C}[x]/x^3)$; we find an object with endomorphisms $k[x]/x^3$ (work out $R$-charge $\implies |x| = 1$).

# Example $k[x]/x^n$

We can resolve $k[x]/x^n$ by compactifying $(\mathbb{A}^2, x^n p)$:

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}(\mathcal{O}(-1)_x \to \mathbb{P}^{1:n}_{q:p}), x^n p).$$




$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^2/\mathbb{Z}_n], x^n) = \langle D^b(\mathbb{A}^1), \ldots, D^b(\mathbb{A}^1)\rangle$$

with the null category at $q = 0$ corresponding to

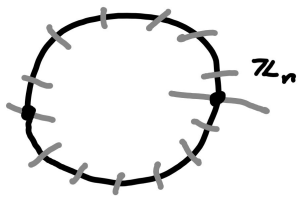
$$\ker \pi_* = \langle D^b_0(\mathbb{A}^1), \ldots, D^b_0(\mathbb{A}^1)\rangle$$

i.e. a collection of $n-1$ spherical objects. (Warning: *not* SOD's)

# Example $k[x]/x^n$

We can resolve $k[x]/x^n$ by compactifying $(\mathbb{A}^2, x^n p)$:

$$\mathscr{C} = \mathrm{MF}(\mathrm{Tot}(\mathcal{O}(-1)_x \to \mathbb{P}^{1:n}_{q:p}), x^n p).$$




$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^2/\mathbb{Z}_n], x^n) = \langle D^b(\mathbb{A}^1), \ldots, D^b(\mathbb{A}^1)\rangle$$

with the null category at $q = 0$ corresponding to

$$\ker \pi_* = \langle D^b_0(\mathbb{A}^1), \ldots, D^b_0(\mathbb{A}^1)\rangle$$

i.e. a collection of $n-1$ spherical objects. (Warning: *not* SOD's)
This is equivalent to the Auslander resolution.

Thank you!