LTCC 2023-2024: Birational Geometry

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Overview

The course will assume the content of Hartshorne[1]. We will look at two main topics which may initially not look hugely related to birational geometry, but in fact encompass a lot of the important ideas.

- Moduli spaces: These are schemes / algebraic spaces / stacks / e.t.c. associated to a collection of algebraic objects (e.g. curves, surfaces, sheaves, ...).
- Foliations: These are a clever way of decomposing a space into subspaces, or analogously considering fibres in a fibre space.

Singularities and adjunction

All schemes are considered over a field k of characteristic 0.

Example. Consider $\mathbb{A}^2 \cong \operatorname{Spec} k[x,y,t]/(xy-t) \xrightarrow{f} \operatorname{Spec} k[t] = \mathbb{A}^1$. For points $p \neq 0$ in $\mathbb{A}^1$, the fibre $f^{-1}(p)$ is a smooth curve (a conic). For $p = 0$ however $f^{-1}(p) = V(xy)$ is a nodal singular curve. What does this say about the moduli space of curves? It says the moduli space of smooth curves is not proper / compact. (One might ask: why care about compactness for moduli spaces?)

Exercise. Show that there does not exist an exact sequence

$$0 \to \mathcal{O}(a) \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}(b) \to 0$$

of sheaves on $\mathbb{P}^2$ for any $a, b \in \mathbb{Z}$. As a consequence, there are no smooth foliations on $\mathbb{P}^2$.

So from looking at the simplest cases of our main topics: the moduli space of curves, and foliations of $\mathbb{P}^2$, we see that singularities naturally arise.

Theorem (Hironaka)**.** *Let X be a smooth variety over k a field of characteristic 0, and let $\mathscr{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then there exists a composition $b : X' \to X$ of blowups in smooth centres (i.e. along subvarieties which are smooth) contained in the singular locus of X, such that the cosupport of $b^{-1}\mathscr{I} = \mathscr{I} \cdot \mathcal{O}_{X'}$ is a simple normal crossings divisor.*

Remark. The cosupport of an ideal sheaf is the support of the quotient; $\operatorname{cosupp}(\mathscr{I}) \coloneqq \operatorname{supp}(\mathcal{O}_X/\mathscr{I})$.

Corollary (Resolution of Singularities)**.** *Let X be a normal variety over k a field of characteristic 0. Then there exists a composition $b : X' \to X$ of blowups in smooth centres contained in the singular locus of X, such that X' is smooth and the exceptional locus $\operatorname{exc}(b)$ is a simple normal crossings divisor. As a consequence, every variety is birational to a smooth variety.*

Proof. Embed $X \hookrightarrow Y$ with Y smooth, which is possible because X is a variety (Chow's lemma). Let $\pi : Y' \to Y$ be the composition of blowups obtained from Hironaka's lemma applied to the subvariety X. Then $\pi^{-1}(X)$ is a divisor, so at some point we must have blown up X. (Since we may assume X has codimension at least 2 in Y.) Since we only blow up in smooth centres, at some point X must have become smooth. □

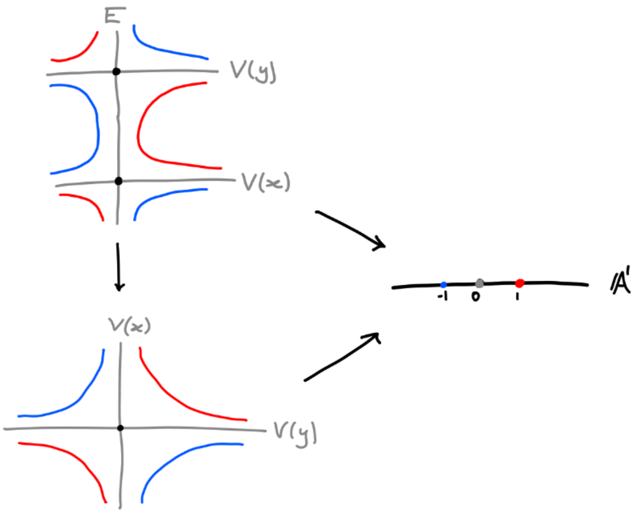
The philosophy of Grothendieck is that we should study morphisms (of schemes e.t.c.), not just the objects (schemes e.t.c.) themselves. Hironaka's theorem tells us that we can make any morphism $X \to \operatorname{Spec} k$ smooth by blowing up. What about general morphisms $f : X \to Z$?[2] The answer is no. Consider the example from above:

$$\mathbb{A}^2 \cong \operatorname{Spec} \frac{k[x,y,t]}{(xy-t)} \to \operatorname{Spec} k[t] = \mathbb{A}^1.$$

[1] With the caveat that he doesn't remember what exactly is and isn't in Hartshorne.

[2] We call a morphism smooth if it is flat and has regular fibres.

This morphism has a singularity at $(0,0)$ in $\mathbb{A}^2$ from the singular fibre $V(xy)$ over $0 \in \mathbb{A}^1$. If we blowup the origin the fibre over $0 \in \mathbb{A}^1$ remains singular because of the exceptional divisor:



So the question becomes: what "nice" singularities should we allow?

Definition. Let $f : (p, X) \to (q, Z)$ be a morphism of germs of smooth varieties. (Think of the germs as $\operatorname{Spec} \mathcal{O}_{X,p}$, or other possibilities.) We say f is *toroidal* if there exist étale / analytic / formal coordinates $x_1, \dots, x_n$ and $z_1, \dots, z_k$ such that f can be written as

$$z_{i_l} = x_1^{m_{1,i_l}} \cdots x_n^{m_{n,i_l}}$$

for some collection of indices $i_l \in \{1, \dots, k\}$. In other words, if f can be written as monomials. We say a morphism $f : X \to Z$ is toroidal if all its germs are toroidal.

Example. Smooth morphisms are toroidal, by the inverse function theorem. The map $\operatorname{Spec} k[x,y,t]/(xy - t) \to \operatorname{Spec} k[t]$ is toroidal, as it can be written in coordinates as $(u,v) \mapsto uv$.

Remark. Toroidal means formally locally isomorphic to a morphism of toric varieties. A good short reference for toric varieties is Fulton's book "Introduction to Toric Varieties".

Conjecture. *Let $f : X \to Z$ be a morphism of varieties. Then there exists a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\alpha} & Z \end{array}$$

such that α and β are birational maps and f' is toroidal.

Remark. We are not assuming that f has connected fibres here. (Recall connected fibres is equivalent to $f_* \mathcal{O}_X = \mathcal{O}_Z$ by Zariski's main theorem.) This is a naive way of generalizing Hironaka's theorem to morphisms, taking the "nice" class of singularities to be the toroidal singularities.

Theorem (Abramovich–Karu)**.** *The above conjecture holds if $f : X \to Z$ is projective and has connected fibres (i.e. $f_* \mathcal{O}_X = \mathcal{O}_Z$).*

Definition. We say $f : X \to Z$ is *semi-stable* if f is toroidal and the scheme-theoretic fibres of f are reduced.

Example. We cannot always achieve semi-stability by blowups; consider again the above example:

$$g : \operatorname{Bl}_{(0,0)} \mathbb{A}^2 \xrightarrow{b} \mathbb{A}^2 \xrightarrow{f} \mathbb{A}^1$$
$$(x,y) \mapsto xy$$

This is a toroidal morphism, but we have

$$\begin{aligned} g^*(0) &= b^*(V(xy)) \\ &= b^*(V(x)) + b^*(V(y)) \\ &= (b_*^{-1}(V(x)) + E) + (b_*^{-1}(V(y)) + E) \\ &= b_*^{-1}(V(x)) + b_*^{-1}(V(y)) + 2E \end{aligned}$$

where E is the exceptional divisor, so $g^{-1}(0)$ is not reduced (having multiplicity 2 along E).

Remark. Semi-stable morphisms "should be" universal families over moduli spaces, whatever that means.

Conjecture (Semi-Stable Reduction)**.** *Let $f : X \to Z$ be a projective morphism with connected fibres. Then there exists a diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{\beta} & X \times_Z Z' & \longrightarrow & X \\ & {\scriptstyle f'}\searrow & \downarrow & & \downarrow{\scriptstyle f} \\ & & Z' & \xrightarrow{\alpha} & Z \end{array}$$

with α generically finite and β birational, such that

1) *X' and Z' are smooth,*

2) *f' is equi-dimensional, and*

3) *f' is semi-stable.*

Remark. If we don't require f' to be smooth this is "weakly semi-stable reduction".

The case $\dim Z = 1$ is known by work of Kempf–Kunetsu–Mumford–Saint Denis. Weakly semi-stable reduction is known in all dimensions by Abramovich–Karu. The case of relative dimension 1 is known by work of de Jong.

Singularities and MMP

Definition. Let X be a smooth variety of dimension n. A *canonical divisor* K_X on X is a Weil divisor such that $\mathcal{O}(K_X) \cong \omega_X := \Omega_X^n$, the sheaf of holomorphic n-forms. If X is a normal variety, with smooth locus $X^{\text{sm}} \subseteq X$ such that $Z = X \setminus X^{\text{sm}}$ has codimension at least 2 in X, then we define a canonical divisor on X as the unique divisor K_X such that $K_X|_{X^{\text{sm}}} = K_{X^{\text{sm}}}$.

Example. If X is a smooth variety and θ is a rational n-form, i.e. a section of $\omega_X \otimes_{\mathcal{O}_X} K(X)$, then

$$K_X = (\theta)_0 - (\theta)_\infty;$$

the locus of zeros minus the locus of poles gives a canonical divisor. In particular, for $\mathbb{P}^n$ we get

$$K_{\mathbb{P}^n} = -(n+1) \cdot H$$

where H is a hyperplane. (Take $dx_0 \wedge \cdots \wedge dx_n$ on $\mathbb{A}^{n+1}$, which extends to a rational n-form on $\mathbb{P}^n$ with a pole of order $n+1$ along the hyperplane at infinity.)

Toric geometry

Definition. The *complex torus* is $(\mathbb{C}^\times)^n = (\mathbb{A}^1 \setminus \{0\})^n$. A *toric variety* is a normal variety X of dimension n which contains the complex torus $(\mathbb{C}^\times)^n$ as a Zariski open subset, such that the action of $(\mathbb{C}^\times)^n$ on itself extends to an action of $(\mathbb{C}^\times)^n$ on X.

Example. We have $\mathbb{A}^n \supseteq (\mathbb{A}^1 \setminus \{0\})^n = (\mathbb{C}^\times)^n$. The torus action is given by $(t_1, \ldots, t_n) \cdots (x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n)$ which extends to all of $\mathbb{A}^n$ and even to $\mathbb{P}^n$. By taking products of tori and their actions we see that $\mathbb{P}^n \times \mathbb{P}^m$ is also toric.

Lemma. *Let X be a toric variety. We have*

$$K_X = - \sum_{D \text{ torus invariant}} D,$$

where a divisor D is torus invariant if for all $t \in (\mathbb{C}^\times)^n$ we have $t \cdot D = D$.

Proof. Look at $\theta = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$ on $(\mathbb{C}^\times)^n$. It has poles along the torus invariant divisors, and no zeros. $\square$

Example. On $\mathbb{A}^n$, the torus invariant divisors are the axes $\{x_i = 0\}$, of which there are n. For $\mathbb{P}^n$ the torus invariant divisors are also the axes $\{x_i = 0\}$, of which there are $n+1$. We again see

$$K_{\mathbb{P}^n} = -\sum_{i=0}^{n} H = -(n+1)H.$$

Exercise. Which quotient singularities are toric? (Recall that a quotient singularity is given by $\mathbb{A}^n/G = \operatorname{Spec} k[x_1, \ldots, x_n]^G$ for G a finite group.)

Canonical divisors and discrepancies

Let X be a smooth variety, and $W \subseteq X$ a smooth subvariety of codimension k. Consider the blowup $b : X' \to X$ in W. What is the relation between K_X and $K_{X'}$? We have the exceptional divisor E, with

$$K_{X'} = b^* K_X + aE.$$

Proposition. *In the above setup, we have $a = k - 1$.*

Proof. This is a local question, and everything is toroidal, and hence the setup is locally isomorphic to the blowup of a toric variety in a torus invariant centre. (Using Artin approximation.) So we are reduced to the case $X' \to \mathbb{A}^n$, where $K_{\mathbb{A}^n} = -\sum\{x_i = 0\}$. Then

$$-K_{X'} = \sum\{x_i = 0\}' + E,$$

where $(\cdot)'$ denotes the strict transform, and

$$-b^* \sum\{x_i = 0\} = \sum\{x_i = 0\}' + kE$$

since W is given by $\{x_1 = \cdots = x_k = 0\}$. $\square$

Exercise. If b is a weighted blowup, what happens to a?

More generally, if $b : X' \to X$ is birational we have

$$K_{X'} = b^* K_X + \sum_{E_i \text{ exceptional}} a(E_i, X) E_i,$$

where $a(E_i, X)$ are called the *discrepancies*. Intuitively, smaller discrepancies mean worse singularities.

Remark. For a general birational morphism, we define the exceptional divisors as the components of the complement of the maximal domain of definition. Equivalently, these are the divisors E in X' such that $b(E)$ is not a divisor in X. The non-existence of exceptional divisors does not imply that the map is an isomorphism; there exist birational morphisms $X' \to X$ with $\dim X = \dim X' = 3$ contracting curves to points.

Example. Consider $C \subseteq \mathbb{P}^2$ a smooth curve of degree d, and $X \subseteq \mathbb{A}^3$ the cone over C. (So $X = \operatorname{Spec} \oplus_{n=0}^{\infty} H^0(C, \mathcal{O}(n)|_C)$, or equivalently $X = \overline{q^{-1}(C)}$ for $q : \mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$.) Consider $b : X' \to X$ the blowup at the cone vertex, with exceptional divisor $E \cong C$. The discrepancy is as follows:

	$a(E,X)$	geometry of X'
$d=1$	1	smooth
$d=2$	0	(Du Val) singularity
$d=3$	-1	more singular
$\vdots$	$\vdots$	$\vdots$

Proof. The degree-genus formula gives

$$\begin{aligned}(d-1)(d-2) - 2 &= 2g - 2 \\ &= K_C \\ &= K_E \\ &= (K_{X'} + E)|_E \qquad \text{by adjunction} \\ &= (b^* K_X + (a+1)E)|_E \\ &= (a+1) \cdot E|_E,\end{aligned}$$

where $b^*K_X \cdot E = 0$ by the projection formula. Now let Y be the blowup of $\mathbb{A}^3$ at the origin, with exceptional divisor $F \cong \mathbb{P}^2$. Then $X' \subseteq Y$ with $F|_{X'} = E$, and $F|_F = \mathcal{O}(-1)$, so we get

$$\begin{aligned} E|_E &= (F|_{X'})|_E \\ &= (F|_F)|_{X'} \\ &= \mathcal{O}(-1)|_{X' \cap F} \\ &= \mathcal{O}(-1)|_E = -d \end{aligned}$$

since $E \cong C$ is a degree d curve in $F \cong \mathbb{P}^2$. Hence $a = 2 - d$. □

Remark. Recall the adjunction formula: If X is a smooth surface, and C is a smooth curve in X, then $(\omega_X \otimes \mathcal{O}(C))|_C \cong \omega_C$. Equivalently $K_C = (K_X + C)|_C$.

Remark. Recall the projection formula: If $f : X \to Y$, with $W \subseteq X$ and L a line bundle on X, then $L|_{f(W)} = f_*((f^*L)|_W)$. Here f_* is the sheaf pushforward, which in the case of connected fibres is the same as the divisor / cycle pushforward. (In the example above this identifies the fiber over ω_X at the origin with the global sections of $b^*\omega_X|_E$, and the former is one-dimensional so $b^*\omega_X|_E$ is trivial. One could also note that $b^*\omega_X|_E$ is the pullback of the line bundle ω_X along the map $E \to X$, which factors through a point.)

Log discrepancies

It is often more natural to work with pairs (X, Δ), for example (X', E) in the above examples of blowups.

Definition. A *log pair* (X, Δ) is a normal variety X together with a $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e. $n(K_X + \Delta)$ is Cartier for some $n > 0$.

Given $b : X' \to X$ birational, we have

$$K_{X'} + b_*^{-1}\Delta + \sum_{E_i \text{ exceptional}} E_i = b^*(K_X + \Delta) + \sum_{E_i \text{ exceptional}} a(E_i, X, \Delta) E_i,$$

where $a(E_i, X, \Delta)$ are the *log discrepancies.*

Definition. A log pair (X, Δ) is *log canonical* if for all birational maps $b : X' \to X$ and all exceptional divisors E_i we have $a(E_i, X, \Delta) \geq 0$. It is *Kawamata log terminal* (KLT) if we have the same with $a(E_i, X, \Delta) > 0$ and Δ has $\mathbb{Q}$-coefficients strictly between 0 and 1.

Remark. In this context "logarithmic" refers to working with boundaries (e.g. Δ). The philosophy is that for D a reduced divisor, the geometry of (X, D) correpsonds to the geometry of $X \setminus D$. For example $H^i(\Omega_X^j(\log D)) = H^{j,i}(X \setminus D)$. (If X is smooth and D is a simple normal crossings divisor then $\Omega_X^1(\log D)$ is analytic locally generated by $dx_1/x_1, \ldots, dx_k/x_k, dx_{k+1}, \ldots, dx_n$ where $D = \{x_1 = \cdots = x_k = 0\}$.)

Example. Consider $(X, \sum_i a_i D_i)$ where X is smooth, and $\sum_i D_i$ is a simple normal crossings divisor.

- If $0 \leq a_i \leq 1$ then this is log canonical.
- If $0 < a_i < 1$ then this is KLT.

Proof sketch. Assume $b : X' \to X$ is the blowup in $W = D_1 \cap \cdots \cap D_k$. Then

$$K_{X'} + \sum_i a_i D_i' + E = b^*\left(K_X + \sum_i a_i D_i\right) + aE,$$

where a is the discrepancy. Now $K_{X'} = b^*K_X + (k-1)E$, and

$$b^*\left(\sum_i a_i D_i\right) = \sum_i a_i D_i' + (a_1 + \cdots + a_k)E,$$

so $a = k - (a_1 + \cdots + a_k)$. □

Example. If X is a toric variety with torus boundary D, then (X, D) is log canonical.

Definition. The *minimal log discrepancy* of a pair (X, Δ) is

$$\mathrm{mld}(X, \Delta) = \inf\{a(E, X, \Delta)\}$$

over all birational morphisms to X and exceptional divisors E.

Remark. The pair is log canonical iff $\mathrm{mld}(X, \Delta) \geq 0$.

Proposition. *If* $\mathrm{mld}(X, \Delta) < 0$*, then* $\mathrm{mld}(X, \Delta) = -\infty$.

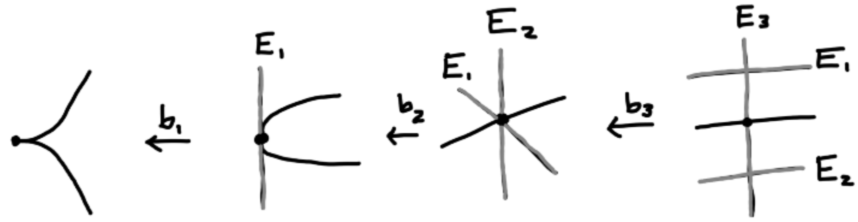
Proof sketch. Consider the local model $(1+\varepsilon)\cdot\{x = 0\}$ in $\mathbb{A}^2$ for $\varepsilon > 0$ under repeated blowups of the origin in the strict transform of $\{x = 0\}$. The pullback of $\{x = 0\}$ is precisely the strict transform plus the new canonical divisor, resulting in discrepancies $1 - n\varepsilon$ where n is the multiplicity of the exceptional divisor in the canonical divisor, which grows without bound. ☐

Log canonical threshold

Definition. For a log pair (X, Δ), the *log canonical threshold* of (X, Δ) with respect to an effective $\mathbb{Q}$-Cartier divisor D is

$$\mathrm{lct}(X, \Delta; D) = \sup\{t : (X, \Delta + t\cdot D) \text{ is log canonical}\}.$$

Example. The pair $(\mathbb{A}^2, \{y^2 = x^3\})$ is not log canonical, which we can see using the following log resolution:



The singularity gives multiplicity two for E_1 when pulling back the curve, giving E_1 and E_2 discrepancy 0 and E_3 discrepancy -1. However for $\frac{5}{6}\{y^2 = x^3\}$ the discrepancies are $\frac{2}{3}$ and $\frac{1}{2}$ for E_1 and E_2, so E_3 gets discrepancy 0 and $(\mathbb{A}^2, \frac{5}{6}\{y^2 = x^3\})$ is log canonical. Hence $\frac{5}{6} \leq \mathrm{lct}(\mathbb{A}^2; \{y^2 = x^3\}) \leq 1$.

Theorem (ACC for LCT; Hacon–McKernan–Xu)**.** *The set*

$$\mathrm{LCT}(n) = \{\mathrm{lct}(X; D) : \dim(X) = n,\ D \textit{ an integral Weil divisor}\}$$

satisfies the ascending chain condition, i.e. if $\lambda_1 \leq \lambda_2 \leq \cdots$ *is a sequence of values in* $\mathrm{LCT}(n)$ *then* $\lambda_i = \lambda_{i-1}$ *for* $i \gg 0$.

Adjunction

Proposition (The adjunction formula)**.** *Let* $D \subseteq X$ *be a smooth divisor in a smooth variety. Then* $\omega_X(D)|_D \cong \omega_D$*, i.e.* $(K_X + D)|_D \sim K_D$.

Proof 1. The short exact sequence $0 \to T_D \to T_X|_D \to N_{D/X} \to 0$ gives $\det T_D \otimes \det N_{D/X} \cong \det T_X|_D$, so $\omega_D^\vee \otimes N_{D/X} \cong \omega_X^\vee|_D$. Then note that $N_{D/X} \cong \mathcal{O}(D)|_D$. ☐

Proof 2. Near a point in D choose coordinates $x_1, \ldots, x_n$ with $D = \{x_1 = 0\}$. Locally, a section of $\omega_X(D)$ is of the form $f\frac{dx_1}{x_1} \wedge dx_2 \wedge \cdots \wedge dx_n$ where $f \in \mathcal{O}_X$. We define the Poincaré residue $\mathrm{Res} : \omega_X(D)|_D \to \omega_D$ by

$$f\frac{dx_1}{x_1} \wedge dx_2 \wedge \cdots \wedge dx_n \mapsto f|_D \cdot dx_2 \wedge \cdots \wedge dx_n.$$

Note that this is in fact a well-defined map of sheaves. Locally, it is surjective since $\alpha \in \omega_D$ lifts to a form $dx_1/x_1 \wedge \alpha$ in $\omega_X(D)|_D$ with residue α, and injective by construction, so it is an isomorphism of sheaves. ☐

If either of X or D is singular then this can fail.

Example. Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, the Hirzebruch surface. We have two rational curves $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1})$ and $\Sigma_{-n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n))$ together with the fibers F satisfying

$$F^2 = 0, \qquad \Sigma_n^2 = n, \qquad \Sigma_{-n}^2 = -n, \qquad \Sigma_n \cdot \Sigma_{-n} = 0, \qquad F \cdot \Sigma_{\pm n} = 1.$$

For $n > 0$, it is a fact that there exists a contraction $p : \mathbb{F}_n \to X$ of Σ_{-n}. Here X turns out to be the cone over the degree n Veronese embedding of $\mathbb{P}^1$. Consider $C = p(F)$. To compute the log discrepancy, suppose

$$p^*(K_X + C) + (a-1)\Sigma_{-n} = K_{\mathbb{F}_n} + F.$$

By the projection formula we have $\Sigma_{-n} \cdot p^*(K_X + C) = 0$, since p contracts Σ_{-n}. On the other hand $\Sigma_{-n} \cdot K_{\mathbb{F}_n} = n - 2$, since $(K_{\mathbb{F}_n} + \Sigma_{-n}) \cdot \Sigma_{-n} = K_{\Sigma_{-n}} = -2$ by adjunction. Hence $n(1-a) = n-1$, so $a = 1/n$. Now $(K_X + C) \cdot C = p^*(K_X + C) \cdot F$ by the projection formula, and so

$$(K_X + C) \cdot C = (K_{\mathbb{F}_n} + F + (1 - 1/n)\Sigma_{-n}) \cdot F = -2 + 0 + (1 - 1/n) \neq -2 = K_C.$$

Hence the adjunction formula fails for C in X. In particular X must be singular.

Theorem. *If X is a normal variety, and $D \subset X$ is a normal reduced divisor, then*

$$(K_X + D)|_D = K_D + \mathrm{Diff}$$

where Diff *is a canonically determined divisor on D, called the* different*, such that*

1. *The different is effective;* $\mathrm{Diff} \geq 0$.
2. *"Adjunction for singularities": If (X, D) is log canonical in a neighbourhood of D, then (D, Diff) is log canonical.*
3. *"Inversion of adjunction": If (D, Diff) is log canonical, then (X, D) is log canonical in a neighbourhood of D.*

Sketch proof. Take a log resolution $p : (X', D') \to (X, D)$. Then

$$p^*(K_X + D) = K_{X'} + D' + \Gamma$$

where Γ is a p-exceptional divisor, and

$$p^*(K_X + D)|_{D'} = (K_{X'} + D' + \Gamma)|_{D'} = K_{D'} + \Gamma|_{D'}$$

so we use $\mathrm{Diff} = p_*(\Gamma|_{D'})$.

1. Proof omitted; uses MMP.
2. If (X, D) is log canonical, let $p : (X', D') \to (X, D)$ be a log resolution. Then

   $$K_{X'} + D' + \sum E_i = p^*(K_X + D) + \sum a_i E_i$$

   gives

   $$K_{D'} + \sum E_i|_{D'} = p^*(K_D + \mathrm{Diff}) + \sum a_i E_i|_{D'},$$

   and $\sum E_i|_{D'} \geq \mathrm{Diff}'$, where Diff' is the strict transform of the different, because the different must have been blown up. Then

   $$K_{D'} + \mathrm{Diff}' + \sum_{E_i \cap \mathrm{Diff}' = \emptyset} E_i|_{D'} = p^*(K_D + \mathrm{Diff}) + \sum a_i E_i|_{D'},$$

   so (D, Diff) is log canonical.
3. This is a corollary of the following theorem:

□

Theorem (Existence of log canonical modifications)**.** *Let (X, Δ) be a log pair. There exists a birational map $p : (X', \Delta') \to (X, \Delta)$, where Δ' is the strict transform of Δ, such that*

1. *$(X', \Delta' + \mathrm{exc}(p))$ is log canonical.*

2. *$K_{X'} + \Delta' + \mathrm{exc}(p)$ is p-ample.*

3. *$K_{X'} + \Delta' + \mathrm{exc}(p) = p^*(K_X + \Delta) + F$ where $-F \geq 0$, and if $F \neq 0$ then every component of Δ' appears with a negative coefficient.*

Remark. The notion "p-ample" in 2 means that there is a divisor G on X such that $K_{X'}+\Delta'+\mathrm{exc}(p)+p^*G$ is ample. Note that 3 is a direct consequence of 2, and implies that the log discrepancies are zero if (X, Δ) is log canonical.

Proof. The proof is via the MMP, see [BCHM]. □

Proof of inversion of adjunction. Consider the case Diff $= 0$. If (X, D) is not log canonical, take a log canonical modification $\pi : (X', D') \to (X, D)$, so

$$K_{X'} + D' + \mathrm{exc}(\pi) = \pi^*(K_X + D) + F$$

with $-F \geq 0$. Since (X, D) is not log canonical we must have $F \neq 0$, so

$$K_{D'} + \mathrm{exc}(\pi)|_{D'} = \pi^*(K_{D'}) + F|_{D'}$$

where $F|_{D'} \neq 0$ because every component of D' appears in F with a negative coefficient. Hence D' is not log canonical. □

Corollary. *Suppose $(X, \Delta) \to T$ is a fibration, where* $\dim T = 1$. *If (X_t, Δ_t) is log canonical, then (X, Δ) is log canonical in a neighbourhood of t.*

Remark. Here a fibration means that $X \to T$ is a fibration (i.e. flat with connected fibres), and for all components D of $\mathrm{Supp}\,\Delta$ the restriction $D \to T$ is also a fibration.

Remark. Hence log canonicity is an open condition on the base.

It is a fact that KLT surface singularities are precisely the quotient singularities. This is false in higher dimensions. There is even a classification of log canonical singularities on surfaces, but in higher dimensions the topology can be arbitrarily complicated.

Exercise. If X is a Calabi-Yau projective variety, show that the cone on X is log canonical.

Moduli of curves

The basic setup in moduli theory is a moduli functor:

$$F : \mathsf{Sch}_{k=\bar{k}} \to \mathsf{Set},$$

where

$$F(k) = \{\text{objects we want to parametrize}\},$$

e.g. curves. We need to decide what $F(T)$ is for general T.
The dream outcome is then

(I) $F \cong_{\Phi} \mathrm{Hom}(-, \mathcal{M})$ for some $\mathcal{M}$, so k-points of $\mathcal{M}$ are the objects we want to classify.

(II) There is a universal family $\mathcal{U} \to \mathcal{M}$ (the distinguished element of $F(\mathcal{M})$) s.t. $s \in F(T)$, $\Phi(s) : T \to M$ corresponds to a pullback family $\mathcal{U} \times_{\mathcal{M}} T \to T$.

For example, for curves $\mathcal{U} \to \mathcal{M}$ should have fiber over a point of $\mathcal{M}$ given by the corresponding curve. To ensure such an $\mathcal{M}$ exists we need to know what a "good" curve is (we can't hope to classify all curves), and also what a "good" *family* of curves is, to define $F(T)$. Since a family of curves is a morphism of relative dimension one, we think of the moduli of curves as being really the moduli of morphisms of relative dimension one.
What "goodness" do we need?

1) A positivity constraint (e.g. something being ample). For curves: $g \geq 2$; ω_C ample.

2) A singularity constraint. (We want $\mathcal{M}$ to be separated and proper.)

So we care about singularities of morphisms.

Definition. A morphism $f : X \to T$ where T is a curve is *locally stable* if it is flat, and $(X, f^{-1}(t))$ is (semi) log-canonical for all $t \in T$.

By adjunction, this is equivalent to the fibres being (semi) log-canonical. (If $\dim(X/T) = 1$ this is equivalent to the fibres being nodal curves.)

Definition. A morphism $f : X \to T$ is *stable* if it is locally stable and K_X is f-ample (i.e. $K_X + f^*D$ is ample for some D).

Semi-stable implies stable, but the converse may fail: $f : \mathbb{A}^3 \to \mathbb{A}^1$, $(x, y, z) \mapsto x^3 + y^3 + z^3$ is locally stable but not semi-stable.

Example. Fix C of genus at least 2. Consider $C \times \mathbb{A}^1 \to \mathbb{A}^1$, and blowup a single point on $C \times \{0\}$. Write E for the exceptional divisor, which is a copy of $\mathbb{P}^1$, and C' for the strict transform of C. We have $(C')^2 = -1$, since $0 = C^2$ so $(C' + E)^2 = 0$, and by Grauert / Artin we can contract C' to a point. The result has a very bad singularity from C', and is no longer locally stable.

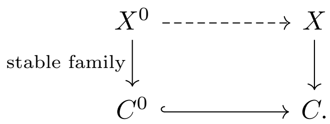
If we allowed this family in our moduli functor, then the point corresponding to $\mathbb{P}^1$ would be in the closure of the point corresponding to C. This would be true for all such C, so $\mathcal{M}$ would be badly non-separated. Hence we need the local stability condition for $\mathcal{M}$ to be separated.
Without the positivity condition, we can do a similar thing repeatedly blowing up points on $C \times \mathbb{A}^1$ to get infinitely many distinct curves in the closure of the point corresponding to C, which would prevent $\mathcal{M}$ from being finite type.

Theorem. *Stable curves with fixed geometric genus form a good moduli space $\overline{\mathcal{M}_g}$.*

(Due to recent work this is true for higher dimensional varieties if you replace genus with volume.)

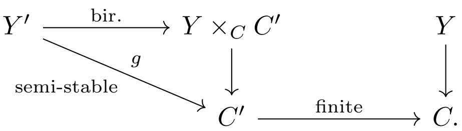
Theorem. *The moduli space of stable curves is separated and proper.*

Proof. For properness, use the valuative criterion:



We want to complete a stable family over $C^0 = C \setminus \{p\}$ to a stable family over C.

- Step 1: Complete X^0 to some family $Y \to C$ (e.g. embed in projective space over C and take closure).
- Step 2: Take a semi-stable reduction of $Y \to C$.



  (Finite base-change is allowed, stack not a scheme.) Then $Y' \to C'$ is locally stable, but $K_{Y'}$ may not be ample over C'.

- Step 3: We want to restore ampleness by contracting all curves E in Y' s.t. $g(E)$ is a point and $K_{Y'} \cdot E \leq 0$. This can be fixed by the MMP.

For separatedness (uniqueness of lifts) we use uniqueness from MMP. □

If C is a stable curve, then $\omega_C^{\otimes 3}$ is very ample, and as an exercise one can check that $h^0(\omega_C^{\otimes 3}) = 5g - 5$. Choosing a basis $\{\sigma_i\}$ for $H^0(\omega_C^{\otimes 3})$ gives an embedding $C \hookrightarrow \mathbb{P}^{5g-6}$, so every stable curve embeds with degree $6g - 6$ in $\mathbb{P}^{5g-6}$. Hence there is a component of the Hilbert scheme H of $\mathbb{P}^{5g-6}$ containing all these curves.

Problem 1. H has points not corresponding to stable curves.

Problem 2. This presentation depends on the choice of basis, so each stable curves appears many times in H. (The stabilizer is $\mathrm{PGL}(5g-5)$.)

So we want a suitable open locus within a suitable quotient $H/\mathrm{PGL}(5g-5)$. Mumford developed "geometric invariant theory" (GIT), which defines a subset $H^s \subseteq H$ of "GIT stable" points, and a Deligne-Mumford stack $H^s /\!\!/ \mathrm{PGL}(5g-5)$ which is a suitable quotient of H by $\mathrm{PGL}(5g-5)$. As a miracle, the set of GIT stable points H^s is precisely the set of points corresponding to stable curves. In higher dimensions this doesn't hold.

Misc

Big and pseff divisors

Recall in the previous section we had a locally stable fibration in curves $X \to C$, and we needed to contract all the curves Σ with $K_X \cdot \Sigma \leq 0$. The following definition is in some sense a weakening of the notion of ample divisors:

Definition. A divisor D is *big* if $mD \sim A + E$ for some $m \geq 0$, where A is ample and E is effective.

Remark. A big divisor is not necessarily numerically effective, e.g. if C is a curve in a surface with $C^2 < 0$, then $(A + nC) \cdot C < 0$ for $n \gg 0$.

Suppose $p : X \to X'$ is a non-trivial birational morphism. If A is ample, then p^*A is not ample, since $p^*A \cdot E = 0$ for each exceptional curve E. (Recall the Nakai–Moishezon criterion: A is ample iff $A^{\dim V} \cdot V > 0$ for all subvarieties V, assuming A is Cartier and the variety is proper. This is equivalent to requiring that $A|_V$ is big for all V.) However, we do get that D big implies p^*D big. In parallel with bigness weakening the notion of ampleness, we have a weakening of the notion of effectiveness:

Definition. A divisor D is *pseudo-effective*, or *pseff*, if $D \cdot A_1 \cdot \dots \cdot A_{\dim X - 1} \geq 0$ whenever $A_1, \dots, A_{\dim X-1}$ are ample.

Remark. A divisor on a curve is pseff iff its degree is non-negative.

Remark. Effective divisors are pseudo-effective, but the converse fails: if p, q are general points on an elliptic curve, then $p - q$ is pseudo-effective but not effective.

Contracting curves

Suppose X is a smooth surface, with K_X pseff, but $K_X \cdot C < 0$ for some curve C.

Claim. $C^2 < 0$.

Proof. Let A be ample. Then $A + mK_X$ is big for $m \gg 0$. (This is not obvious, but big divisor plus a pseff divisor is big.) Hence $A + mK_X \sim_{\mathbb{Q}} A' + E'$ for $m \gg 0$, where A' is $\mathbb{Q}$-ample and E' is $\mathbb{Q}$-effective, which gives

$$E' \cdot C < (A' + E') \cdot C = (A + mK_X) \cdot C < 0.$$

This is only possible if C is in the support of E' with $C^2 < 0$. □

Note that from adjunction we have $K_C + \mathrm{Diff} = (K_X + C) \cdot C < 0$, so K_C is negative meaning C must be $\mathbb{P}^1$, with $K_C = -2$. (A priori C is singular, but since $\mathbb{P}^1$ is smooth we see that $\mathrm{Diff} = 0$ and classical adjunction holds.)

Note that if $b : S' \to S$ is the blowup of a surface at a smooth point with exceptional divisor C, then $C^2 = K_{S'} \cdot C = -1$.

Theorem (Castelnuovo's criterion)**.** *Suppose X is a surface, with a curve C such that $C^2 = K_X \cdot C = -1$. Then there exists a blowup $b : X \to Y$ at a smooth point of Y with exceptional divisor C.*

Proof. Let $\widehat{X}$ be the formal completion of X along C, i.e. the ringed space $(\mathbb{P}^1, \varprojlim_n \mathcal{O}_X/\mathcal{O}_C(-nC))$. If X_n is the nth infinitesimal neighbourhood, we have a SES

$$0 \to \mathcal{O}_C(-nC) \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}} \to 0,$$

and $\mathcal{O}_C(-nC) = \mathcal{O}_{\mathbb{P}^1}(n)$ since $C = \mathbb{P}^1$ as argued above. Now $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$ for $n \geq 0$, so the SES splits, and hence

$$\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \Gamma(\mathbb{P}^1, \prod_{n \geq 0} \mathcal{O}_{\mathbb{P}^1}(n)) = \mathbb{C}[\![x, y]\!].$$

So we get $\widehat{X} \to \widehat{\mathbb{A}^2} = \operatorname{Spf} \mathbb{C}[\![x, y]\!]$. In fact $\widehat{X} \to \widehat{\mathbb{A}^2}$ is the completion of the blowup of $\mathbb{A}^2$ at a point. By Artin approximation this extends to a morphism $X \to Y$ where Y is smooth. We can push an ample line bundle on X forward to see that Y is projective. □

Remark. Every smooth 2-dimensional algebraic space is projective, which follows from MMP.

MMP for surfaces

Suppose X is a smooth surface, and K_X is pseff. The MMP runs as follows: if there is a curve C with $K_X \cdot C < 0$, then by Castelnuovo's criterion we can contract it. Otherwise we stop, and X is the minimal model. (K_X may not be ample.)
Since $\dim H_2(X, \mathbb{R})$ drops with each contraction, the algorithm terminates. (Note that $[C]$ is non-zero in $H_2(X, \mathbb{R})$ since $C \cdot K_X \neq 0$.)

Remark. If K_X is not pseff we can do the same thing, instead terminating in a Mori fiber space (for surfaces this is $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle.) The challenge is finding which curves are (-1)-curves.

We write $X \to X_{\text{min}}$ for the minimal model.

Remark. If K_X is pseff, then $K_{X_{\text{min}}}$ is nef ($K_{X_{\text{min}}} \cdot C \geq 0$ for all C.) If K_X is not pseff then running the algorithm always results in a Mori fiber space.

Definition. A Mori fiber space is a space X with a morphism $f : X \to Y$ such that $f_* \mathcal{O}_X = \mathcal{O}_Y$, $\dim X > \dim Y$, $\rho(X) - \rho(Y) = 1$ and $-K_X$ is f-ample.

Theorem. *If $f : X \to Y$ is a projective morphism, and A is ample on Y, then f^*A is nef.*

Remark. $K_{X_{\text{min}}}$ is not necessarily ample.

Theorem. *If X is a smooth surface, K_X nef implies K_X is semi-ample, i.e. there is a morphism $f : X \to Y$ with $K_X \sim_{\mathbb{Q}} f^*A$ for some ample A.*

If Y is a point then $K_X \sim \mathbb{Q}0$, and if Y is a curve then $X \to Y$ is an elliptic fibration. If Y is a surface we call Y the canonical model X_{can}.

Remark. If K_X is pseff but not big, we get $\dim Y \leq 1$. If K_X is big we get a canonical model.

The point of canonical models is that $K_{X_{\text{can}}}$ is ample. We can then apply this for showing that the moduli space of curves is proper. Recall we had a family $X^0 \to C^0 = C \setminus \{0\}$ to extend over C.

1. Complete to get $\overline{X} \to C$.
2. Apply semi-stable reduction to get $X' \to C$.
3. Go to a canonical model $X' \to X \to C$.

Then $X \to C$ is the family we want.

Remark. X_{can} is usually singular, but has mild (canonical) singularities, e.g. Du Val singularities.

Remark. For surfaces, the singularities of X_{can} are analytically locally of the form $\mathbb{C}^2/G$ with $G \leq \mathrm{SL}_2(\mathbb{C})$.