Matrix Factorizations and Knörrer Periodicity

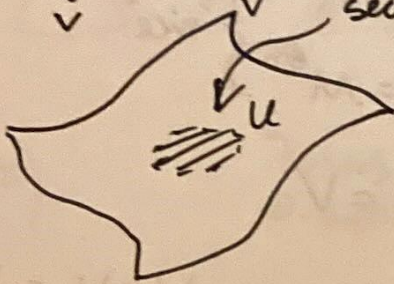
Sheaves on varieties:

maps to $\mathbb{P}^n$ — maps to $Gr(k,n)$ (or vector bundle)

(e.g. 1) $\mathcal{L}$ line bundle

$\downarrow$

V — sections of $\mathcal{L}|_U$: sheaf $\mathcal{L}$

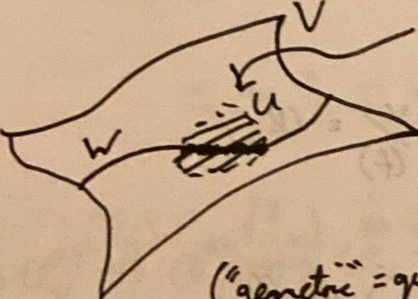


"pushforward"

(or sections of sheaf $\mathcal{F}$ on W)

(e.g. 2) $W \subset V$ subvariety

functions on $W \cap U$: sheaf $\mathcal{O}_W$



("geometric" = quasi-coherent)

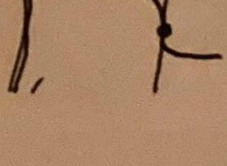
Locally: Sheaf / affine V $\longleftrightarrow$ Module / ring $\mathcal{O}_V$

- $W = V \cap \{f_1 = \cdots = f_n = 0\} \longleftrightarrow \mathcal{O}_W = \mathcal{O}_V / (f_1, \ldots, f_n)$
- trivial rank n bundle $\mathbb{A}^n \times V \longleftrightarrow \mathcal{O}_V^{\oplus n}$ free module

$\mathcal{O}_W$ like $\mathbb{1}_W$: $\dim \mathcal{O}_W|_p = \begin{cases} 0 & p \notin W \\ 1 & p \in W \end{cases}$

$\Rightarrow \mathcal{O}_{W_1} \otimes \mathcal{O}_{W_2} = \mathcal{O}_{W_1 \cap W_2} \qquad (\mathbb{1}_A \cdot \mathbb{1}_B = \mathbb{1}_{A \cap B})$

"bad" intersections $\longleftrightarrow$ "bad" tensor products



$\Rightarrow$ understand intersection degeneracy using $\mathrm{Tor}^*(-,-)$

Definition of Tor: replace module with ~~resolution~~ by modules with "nice" tensor products (flat)

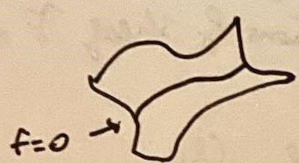
$\mathrm{Tor}^0 = \otimes$

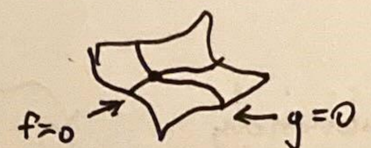
$\mathrm{Tor}^{>0}$ = degeneracy of $\otimes$

e.g. free modules

$$\left.\begin{array}{l} R \otimes_R M = M \\ R^{\oplus n} \otimes_R M = M^{\oplus n} \end{array}\right\} \text{"nice"}$$

e.g. $W_1 = \{f=0\} \subset V$, $W_2 = \{fg = 0\} \subset V$





$W_1 \cap W_2 = W_1$ degenerate

free resolution: $\left[\mathcal{O}_V \xrightarrow{f} \mathcal{O}_V\right] \to \mathcal{O}_V/(f) = \mathcal{O}_{W_1}$

$$\Rightarrow \mathrm{Tor}^*(\mathcal{O}_{W_1}, \mathcal{O}_{W_2}) = (\mathcal{O}_V \xrightarrow{f} \mathcal{O}_V) \otimes \mathcal{O}_V/(f \cdot g)$$

$$= \mathcal{O}_V/(f\cdot g) \xrightarrow{f} \mathcal{O}_V/(f\cdot g)$$

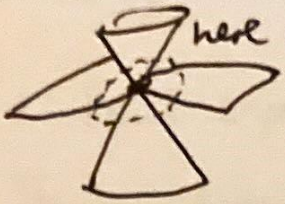
H_1: $(g) = \mathcal{I}^{W_1}_{W_2}$ $\qquad$ H_0: $\mathcal{O}_V/(f) = \mathcal{O}_{W_1}$

intersection: $\mathcal{O}_{W_1}$

degeneracy: $\mathcal{I}^{W_1}_{W_2}$

Moral: free resolutions encode degeneracy behaviour of a sheaf/module.

Dichotomy: singularities vs. smooth geometry

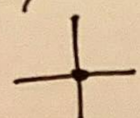


singularities $\rightsquigarrow$ want to study locally ($\Rightarrow$ affine)

smooth geometry $\rightsquigarrow$ only interesting globally ($\Rightarrow$ projective)

(Serre 1955) $\underbrace{\exists \text{ sheaves with only } \infty\text{-length (locally) free resolutions}}$ $\qquad$ $\forall$ (f.g.) sheaves $\exists$ (locally) free resolution of length $\leq \dim X$

e.g. 1



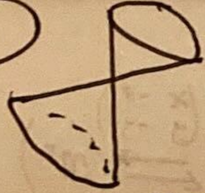
$V = \{xy = 0\} \subset \mathbb{A}^2, \quad \mathcal{O}_V = \mathbb{C}[x,y]/(xy)$

$$\cdots \xrightarrow{x} \mathcal{O}_V \xrightarrow{y} \mathcal{O}_V \xrightarrow{x} \mathcal{O}_V \to \mathcal{O}_V/(x) \to 0$$

$$\underbrace{\cdots \to \mathcal{O}_V^{\oplus 2} \xrightarrow[N]{\begin{pmatrix} x & \\ & y \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow[M]{\begin{pmatrix} y & \\ & x \end{pmatrix}}}_{\text{(eventually) } \mathbb{Z}\text{-periodic}} \mathcal{O}_V^{\oplus 2} \xrightarrow{(x \ y)} \mathcal{O}_V \to \mathcal{O}_V/(x,y) \to 0$$

$$\left[M \cdot N = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix} = N \cdot M \right.$$

e.g. 2



$V = \{xy = z^2\} \subset \mathbb{A}^3, \quad \mathcal{O}_V = \mathbb{C}[x,y,z]/(xy - z^2)$

$$\cdots \xrightarrow{\begin{pmatrix} y & z \\ -z & -x \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow[N]{\begin{pmatrix} x & z \\ -z & -y \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow[M]{\begin{pmatrix} y & z \\ -z & -x \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow{(x \ z)} \mathcal{O}_V \to \mathcal{O}_V/(x,z) \to 0$$

$$M \cdot N = \begin{pmatrix} xy - z^2 & 0 \\ 0 & xy - z^2 \end{pmatrix} = N \cdot M \qquad \underline{\text{matrix factorization}}$$

(Eisenbud 1980) Any (f.g.) module / affine hypersurface singularity $\subset \mathbb{A}^n$ has an <u>eventually</u> $\mathbb{Z}$-periodic free resolution given by a matrix factorization.

$\Big[$ $MF(f) \simeq$ {modules w/ <u>immediately</u> $\mathbb{Z}$-periodic resolutions}

$= MCM$, stabilization $\Big]$

<u>Fact:</u> $\mathrm{Hom}(M,N) = \{\text{chain maps } P_M^\bullet \to P_N^\bullet\} \big/ (\text{chain homotopy})$

free resolutions

$\Rightarrow$ define category $MF(\mathbb{A}^n, f)$:

- objects $\mathcal{E} = \left(\mathcal{O}_{\mathbb{A}^n}^{\oplus k} \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}_{\mathbb{A}^n}^{\oplus k}\right)$ s.t. $MN = NM = f \cdot I$
- morphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \{\text{chain maps } \mathcal{E} \to \mathcal{F}\text{, 2-periodic}\} \big/ (\text{chain homotopy, 2-periodic})$

Getting back:

$$\left(\mathcal{O}_{\mathbb{A}^n}^{\oplus k} \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}_{\mathbb{A}^n}^{\oplus k}\right) \rightsquigarrow \cdots \to \mathcal{O}_V^k \xrightarrow{M} \mathcal{O}_V^k \xrightarrow{N} \mathcal{O}_V^k \xrightarrow{M} \mathcal{O}_V^k \to K \to 0$$

K MCM

$$MF(\mathbb{A}^n, f) \underset{\sim}{\xrightarrow{\mathrm{coker}(M)}} MCM(\mathcal{O}_V) \simeq \{\text{sheaves}/V\} \big/ \{\text{sheaves w/ finite resolutions}\}$$

<u>Fact:</u> Equivalence (in particular, $\mathrm{coker}(M)$ free $\Rightarrow$ $\mathcal{E}$ null-homotopic)

e.g. $\mathcal{E} = \left(\mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}\right) \in MF(\mathbb{A}^2, xy)$.

$$\mathcal{H}om(\mathcal{E},\mathcal{E}) = \mathcal{E}^\vee \otimes \mathcal{E} = \left(\mathcal{O} \underset{-x}{\overset{y}{\rightleftarrows}} \mathcal{O}\right) \otimes \left(\mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}\right) = \left(\mathcal{O}^2 \underset{\begin{pmatrix} y & -x \\ y & -x \end{pmatrix}}{\overset{\begin{pmatrix} x & -x \\ y & -y \end{pmatrix}}{\rightleftarrows}} \mathcal{O}^2\right)$$

2-periodic chain complex

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) = H^0(\mathcal{H}om(\mathcal{E},\mathcal{E})) = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \big/ \langle \begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix} \rangle = \mathcal{O}_{\mathbb{A}^2} \big/ (x,y) = \mathbb{C}.$$

$$\left(H^1 = \langle \begin{pmatrix} x \\ y \end{pmatrix} \rangle \big/ \langle \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0 \right).$$

Orlov 2004 $= D_{Sg}(V)$; $MF(f) \rightsquigarrow \mathrm{Sing}(V)$

e.g. "supported" on $\mathrm{Crit}(f)$

$$\partial_x(f) = \partial_x(M \cdot N) = \underbrace{\partial_x(M) \cdot N + M \cdot \partial_x(N)}_{\text{null-homotopy}}$$

Knörrer Periodicity: Wanted to ~~study~~ relate $f(x,y)+z_1^2+\cdots+z_n^2=0$

(1987) to $f(x,y)=0$ ("simple" hypersurface singularities)

Found 2-periodicity and near-1-periodicity of MCMs $\leftrightarrow$ MFs.

Thm: $MF(\mathbb{A}^n, f) \simeq MF(\mathbb{A}^{n+2}, f+xy)$

(classical K.P.)

(equiv. $z^2+w^2 = \overbrace{(z-iw)}^{x}\overbrace{(z+iw)}^{y}$)

[Functors: $\overset{-\otimes\mathcal{E}}{\longrightarrow}$, $\longleftarrow$ $\mathcal{H}om(\mathcal{E},-)$ + "pushforward"

where $\mathcal{E} = (\mathcal{O} \overset{x}{\underset{y}{\rightleftarrows}} \mathcal{O}) \in MF(\mathbb{A}^2, xy)$]

e.g. $MF(\mathbb{A}^3, xy-z^2) \simeq MF(\mathbb{A}^1, -z^2) = \langle \overbrace{(\mathcal{O} \overset{z}{\underset{-z}{\rightleftarrows}} \mathcal{O})}^{\mathcal{Z}} \rangle$

$\cancel{\mathcal{H}om(\mathcal{E},\mathcal{Z})} = \cancel{\mathcal{E}^\vee \otimes \mathcal{Z}}$

$\cancel{= (\mathcal{O} \overset{y}{\underset{-x}{\rightleftarrows}} \mathcal{O}) \otimes (\mathcal{O} \overset{z}{\underset{-z}{\rightleftarrows}} \mathcal{O})}$

$\cancel{= \left(\mathcal{O}^2 \overset{\begin{pmatrix} z & -x \\ y & z\end{pmatrix}}{\underset{\begin{pmatrix} -z & -x \\ y & -z\end{pmatrix}}{\rightleftarrows}} \mathcal{O}^2\right)}$

$\mathcal{E} \otimes \mathcal{Z} = (\mathcal{O} \overset{x}{\underset{y}{\rightleftarrows}} \mathcal{O}) \otimes (\mathcal{O} \overset{z}{\underset{-z}{\rightleftarrows}} \mathcal{O})$

$= \left(\mathcal{O}^2 \overset{\begin{pmatrix} z & y \\ x & z\end{pmatrix}}{\underset{\begin{pmatrix} -z & y \\ x & -z\end{pmatrix}}{\rightleftarrows}} \mathcal{O}^2\right)$

seen before: stabilization of $\mathcal{O}_{\{x=z=0\}}$.

Constant/trivial family version of:

(proto K.P.) **Prop:** $MF(\mathbb{A}^2, xy) \underset{\mathcal{H}om(\mathcal{E},-)}{\overset{-\otimes\mathcal{E}}{\simeq}} MF(\mathbb{C}, 0)$. $\leftarrow$ (2-periodic) complexes of $\mathbb{C}$-v.s. mod homotopy $\simeq \{\mathbb{C}^n \oplus \mathbb{C}^m[1]\}$

proof: Composite $\circlearrowleft MF(\mathbb{C}, 0) = \langle \mathbb{C} \rangle$

$\mathbb{C} \longmapsto \mathcal{H}om(\mathcal{E},\mathcal{E}) \simeq \underbrace{\mathbb{C}}_{H^0} \oplus \underbrace{0[1]}_{H^1}$ computed earlier.

$\Rightarrow$ identity functor.

Other composite: $\quad MF(\mathbb{A}^2, xy) \circlearrowleft$

$$\mathcal{F} \longmapsto \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} \qquad \left[\cancel{\ } \ \mathcal{E}^{h^0} \oplus \mathcal{E}[1]^{h^1} \right]$$

<u>Fact</u>: have "exact sequence"

$$\mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} \xrightarrow{ev.} \mathcal{F} \longrightarrow \mathrm{Cone},$$

and ev. is an iso. $\Leftrightarrow$ $\mathrm{Cone} \simeq 0$.

Apply $\mathcal{H}om(\mathcal{E}, -)$:

$$\begin{tikzcd} \mathcal{H}om(\mathcal{E},\mathcal{F}) \otimes_{\mathbb{C}} \underbrace{\cancel{\mathcal{H}om(\mathcal{E},\mathcal{E})}}_{=\mathbb{C}} \arrow[r] \arrow[r, bend right, "iso."'] & \mathcal{H}om(\mathcal{E},\mathcal{F}) \arrow[r] & \mathcal{H}om(\mathcal{E}, \mathrm{Cone}) \end{tikzcd}$$

$$\Rightarrow \mathcal{H}om(\mathcal{E}, \mathrm{Cone}) = 0.$$

<u>Goal</u>: $\mathcal{H}om(\mathcal{E}, \mathrm{Cone}) = 0 \Rightarrow \mathrm{Cone} \simeq 0$ (i.e. null-homotopic).
(i.e. $H_0 = H_1 = 0$)

$$\mathcal{H}om\left(\mathcal{E}, \mathcal{O}^k \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}^k\right) = \left(\mathcal{O} \underset{-x}{\overset{y}{\rightleftarrows}} \mathcal{O}\right) \otimes \left(\mathcal{O}^k \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}^k\right)$$

$$= \left(\mathcal{O}^{2k} \underset{\begin{pmatrix} N & -x \\ y & -M \end{pmatrix}}{\overset{\begin{pmatrix} M & -x \\ y & -N \end{pmatrix}}{\rightleftarrows}} \mathcal{O}^{2k} \right)$$

$$H^0 = \left\{ (v,w) : \begin{matrix} Mv = xw \\ \Updownarrow \\ Nw = yv \end{matrix} \right\} \Big/ \left\langle \begin{pmatrix} N \\ y \end{pmatrix}, \begin{pmatrix} x \\ M \end{pmatrix} \right\rangle$$

$$H^1 = \left\{ (v,w) : \begin{matrix} Nv = xw \\ \Updownarrow \\ Mw = yv \end{matrix} \right\} \Big/ \left\langle \begin{pmatrix} M \\ y \end{pmatrix}, \begin{pmatrix} x \\ N \end{pmatrix} \right\rangle$$

$$\text{so } H^0 = H^1 = 0 \Rightarrow \begin{cases} x \mid Mv \Rightarrow v \in \operatorname{im} N + (x) \\ y \mid Nw \Rightarrow w \in \operatorname{im} M + (y) \\ x \mid Nv \Rightarrow v \in \operatorname{im} M + (x) \\ y \mid Mw \Rightarrow w \in \operatorname{im} N + (y) \end{cases}$$

$\Rightarrow \left(\mathcal{O}^k \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}^k\right)$ is exact mod (x) and mod (y).

$$\Rightarrow \operatorname{coker} M = \mathcal{O}^k / \operatorname{im} M$$
$$= \mathcal{O}^k / \ker N \quad \text{exact}$$
$$= \operatorname{im} N$$
$$= \ker M \quad \text{exact}$$
$$= \text{free module} \quad (\text{structure thm for } \mathbb{C}[x] / \mathbb{C}[y])$$
$$(\text{mod}(x) \text{ and mod}(y).)$$

$\Rightarrow \operatorname{coker} M$ is a free module / $\mathbb{C}[x,y]/(xy)$

$\Rightarrow \left(\mathcal{O}^k \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{O}^k\right)$ null-homotopic.

$\blacksquare$

Outlook: Non-trivial family version:

$$MF(X \times \mathbb{A}^1, f(x)y) \simeq MF(\underset{\{f(x)=0\} \subset X}{\overset{\shortparallel}{Y}}, 0).$$

Study singular Y using smooth $X \times \mathbb{A}^1$ with function $f(x)y$.

Not just hypersurfaces:

$$MF(f_1(x)y_1 + \cdots + f_n(x)y_n) \simeq MF(\{f_1(x) = \cdots = f_n(x) = 0\}, 0)$$

complete intersection

Can go from $\mathbb{Z}/2$ to $\mathbb{Z}$-grading w/ extra data

$$\rightsquigarrow MF(X,0) \simeq D^b(X)$$

Can do non-affine base (subtler defn's)

sheafy MF's

e.g. $\mathcal{E} \simeq \left(\mathcal{O}/x \rightleftarrows 0\right)$. $\mathcal{H}om(\mathcal{E}, \mathcal{E}) = \mathcal{H}om\left(\mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}, \mathcal{O}/x\right)$

$$= \left(\mathcal{O}/x \xrightarrow{y} \mathcal{O}/x\right) \simeq \mathcal{O}/(x,y) = \mathbb{C}.$$