Foliations of Algebraic Surfaces

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0 Introduction

Foliations have had famous applications in the realm of real differential geometry, due to the work of Thurston and his geometrization conjecture. However, holomorphic foliations in complex geometry have also been a fruitful area of study. Here we will specifically look at the applications of foliations to the study of complex algebraic surfaces, and their birational geometry. In analogy with the Enriques classification of surfaces and the Minimal Model Program, foliations of surfaces also have a rich classification theory [Br]. This report is an introduction to the basic concepts of the theory of foliations on algebraic surfaces, referencing important results but with many omissions, including almost all of the classification theory. We have attempted to motivate the concepts with lots of basic examples, and judicious use of pictures.

The structure of the report is as follows. We begin with an introduction to the basic concepts of foliations in Section 1, and the local analysis of singularities, mentioning the Camacho–Sad separatrix theorem, and the behaviour of foliations under blowup including Seidenberg's theorem. Section 2 covers index theorems giving some of the intersection theory of the canonical bundle of a foliation, which we then apply in Section 3 to some examples of foliations arising from fibrations. Finally, in Section 4 we make some remarks about the work of Bogomolov and McQuillan towards the Green–Griffiths conjecture, before going on a tenuously related tangent about slope stability and the Donaldson–Uhlenbeck–Yau theorem.

1 Foliations and blowups

In this section we cover the basic definitions of foliations, and introduce some of their local and global invariants, before looking at their behaviour under blowups. Everything will be considered over $\mathbb{C}$.

1.1 Foliations

Suppose X is a normal surface. To give a foliation $\mathscr{F}$ of X, we want to define the tangent direction of the leaves of the foliation at any point in X, giving a tangent vector up to scale at every point. We can accomplish this by taking vector fields $v_i \in H^0(U_i, T_X)$ for an open cover $X = \cup_i U_i$, such that $v_j = f_{ij} v_i$ for some non-vanishing holomorphic function $f_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_X^\times)$. Identifying data which would give rise to the same foliation, we see that $\mathscr{F}$ is uniquely determined by the class $\{f_{ij}\}$ in Čech cohomology $H^1(X, \mathcal{O}_X^\times)$, defining a line bundle $T_\mathscr{F}$, and the global section $\{v_i\}$ of $T_\mathscr{F}^\vee \otimes T_X$ up to multiplication by a nowhere vanishing holomorphic function.

Example 1. Consider foliating $\mathbb{A}^2$ with horizontal leaves. The tangent direction for the leaves is generated by $\frac{\partial}{\partial x}$, so $T_\mathscr{F}$ is trivial (as it must be on $\mathbb{A}^2$) with the section of $T_\mathscr{F}^\vee \otimes T_{\mathbb{A}^2} = T_{\mathbb{A}^2}$ given by $\frac{\partial}{\partial x}$. Replacing $\frac{\partial}{\partial x}$ by $-\frac{\partial}{\partial x}$ or $2\frac{\partial}{\partial x}$ defines the same foliation.

Figure 1: Foliation of $\mathbb{A}^2$ generated by $\frac{\partial}{\partial x}$.

We can also characterize the tangent spaces to the leaves as kernels of 1-forms, leading to a similar definition with T_X replaced by Ω_X^1. The foliation is given by a line bundle $N_\mathscr{F}^\vee$ and a global section $\{\omega_i\}$ of $N_\mathscr{F} \otimes \Omega_X^1$ up to multiplication by a nowhere vanishing holomorphic function.

Example 2. The tangent vector $\frac{\partial}{\partial x}$ on $\mathbb{A}^2$ is annhilated by the 1-form dy, and so the foliation from the previous example can also be seen as given by the trivial bundle $N_\mathscr{F}$ and the section dy of $N_\mathscr{F} \otimes \Omega_{\mathbb{A}^2}^1$. We could construct dy here by contracting $\frac{\partial}{\partial x}$ with the area form $dx \wedge dy$, and so more generally the foliation generated by the vector field $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ could also be specified by the contracted 1-form $Bdy - Adx$.

For a smooth foliation the tangent bundle $T_\mathscr{F}$ should be locally free, so we would require that the vector fields $\{v_i\}$ (or the 1-forms $\{\omega_i\}$) be non-vanishing. This is quite restrictive, and so in practice we will be interested in foliations with singularities. We can also view the above definitions as giving rational sections of T_X or Ω_X^1, defining a global section after twisting by $\mathcal{O}(D)$ for D a suitable divisor of poles.

There is a unique choice of D such that the resulting section vanishes only in codimension 2, and so we restrict our attention to foliations with singularities in codimension 2.

Example 3. Consider the extension of the foliation generated by $\frac{\partial}{\partial x}$ on $\mathbb{A}^2$ to $\mathbb{P}^2$. In coordinates $[1 : y : z]$ the 1-form dy on $\mathbb{A}^2$ becomes $d(y/z) = dy/z - ydz/z^2$, with a pole of order 2 along the hyperplane at infinity. Hence we get a section $(zdy - ydz)/z^2$ of $\Omega^1_{\mathbb{P}^1}(2)$ which vanishes at $[1 : 0 : 0] \in \mathbb{P}^2$, defining a foliation $\mathscr{F}$ with $N_{\mathscr{F}} = \mathcal{O}(2)$ and a singularity at $[1 : 0 : 0]$. Note that the local form of the singularity is generated by the radial vector field $y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ annhilated by $zdy - ydz$, illustrated in Fig. 2.

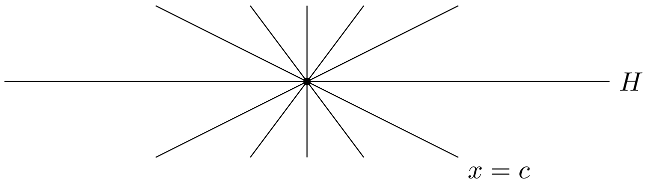

Figure 2: Parallel leaves converging at the horizon.

The perspective of a foliation as the subsheaf $T_{\mathscr{F}}$ in T_X leads us to a general definition of foliations for higher dimensions.

Definition 1.1. A rank r foliation $\mathscr{F}$ of a normal variety X is a rank r subsheaf $T_{\mathscr{F}}$ of T_X such that

i. $T_X/T_{\mathscr{F}}$ is torsion-free,

ii. $T_{\mathscr{F}}$ is closed under the Lie bracket on T_X.

The singular locus $\mathrm{Sing}(\mathscr{F})$ is the singular locus of X together with the locus where the fiber of $T_{\mathscr{F}} \to T_X$ drops in rank.

Remark. Note that ii. is automatically satisfied for rank 1 foliations, as the Lie bracket then vanishes on $T_{\mathscr{F}}$. This condition is the requirement for integrability, leading to Theorem 1.5.

See [Fr, §2] for properties of torsion-free sheaves and singularities of coherent sheaves. It follows that $T_X/T_{\mathscr{F}}$ is a subsheaf of a locally free sheaf of the same rank, so in the case of a rank 1 foliation on a surface we get $T_X/T_{\mathscr{F}} = I \cdot N_{\mathscr{F}}$ where $N_{\mathscr{F}}$ is a line bundle and I is the ideal sheaf cutting out $\mathrm{Sing}(\mathscr{F})$. We then have two dual exact sequences

$$0 \to T_{\mathscr{F}} \to T_X \to I \cdot N_{\mathscr{F}} \to 0 \tag{1}$$

and

$$0 \to N^\vee_{\mathscr{F}} \to \Omega^1_X \to I \cdot T^\vee_{\mathscr{F}} \to 0. \tag{2}$$

Of course, the 1-forms vanishing on $T_{\mathscr{F}}$ are dual to $T_X/T_{\mathscr{F}}$, justifying the re-use of the name $N_{\mathscr{F}}$ from above. The inclusion $N^\vee_{\mathscr{F}} \to \Omega^1_X$ is given by the section of $N_{\mathscr{F}} \otimes \Omega^1_X$ defining the foliation.

Remark. Since a foliation is specified by an (almost) non-vanishing vector field up to scale, we can view it as an (almost) section of the projective tangent bundle $\mathbb{P}(T_X) \to X$, i.e. a rational section with domain of definition the complement of $\mathrm{Sing}(\mathscr{F})$. Similarly, the 1-form perspective gives a rational section of the projective cotangent bundle $\mathbb{P}(\Omega^1_X) \to X$. The fact that these are equivalent can be seen from the fact that $\Omega^1_X = T_X \otimes K_X$, giving an isomorphism $\mathbb{P}(\Omega^1_X) = \mathbb{P}(T_X)$. See [M98, II] for more context. This will come up in Section 4.

We now restrict our attention to the case of a rank 1 foliation $\mathscr{F}$ on a surface X.

Proposition 1.2. $K_X = T^\vee_{\mathscr{F}} \otimes N^\vee_{\mathscr{F}}$.

Proof. Non-vanishing sections of K_X give by contraction an isomorphism between vector fields and 1-forms, identifying $T_{\mathscr{F}}$ and $N^\vee_{\mathscr{F}}$. In this way we get an isomorphism $K_X = \mathcal{H}om(T_{\mathscr{F}}, N^\vee_{\mathscr{F}}) = T^\vee_{\mathscr{F}} \otimes N^\vee_{\mathscr{F}}$. This can also be seen from the exact sequence Eq. (2), which restricts to an exact sequence of vector bundles on $U = X \setminus \mathrm{Sing}(\mathscr{F})$, giving $K_X|_U = (N^\vee_{\mathscr{F}} \otimes T^\vee_{\mathscr{F}})|_U$ by taking determinants, which implies the result because $\mathrm{Sing}(\mathscr{F})$ has codimension at least 2. □

Example 4. In the example of $\frac{\partial}{\partial x}$ extended to $\mathbb{P}^2$ we had $N_{\mathscr{F}} = \mathcal{O}(2)$. Since $K_{\mathbb{P}^2} = \mathcal{O}(-3)$, the proposition gives $T_{\mathscr{F}} = \mathcal{O}(1)$, so $\frac{\partial}{\partial x}$ should extend to a section of $T_{\mathbb{P}^2}(-1)$. Indeed in coordinates $[1 : y : z]$ the pushforward of $\frac{\partial}{\partial x}$ is $-z(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$, vanishing to first order at infinity.

Definition 1.3. The *canonical bundle* of the foliation $\mathscr{F}$ is $K_{\mathscr{F}} := K_X \otimes N_{\mathscr{F}} = T_{\mathscr{F}}^{\vee}$.

Example 5. In the previous example $T_{\mathscr{F}} = \mathcal{O}(1)$, so $K_{\mathscr{F}} = \mathcal{O}(-1)$.

1.2 Invariant curves

To relate a foliation defined by a vector field or 1-form to the intuitive notion of leaves foliating a surface, we have to consider its integral curves, which give the leaves of the foliation. In general, the solutions to the relevant differential equation will be analytic but not algebraic.

Definition 1.4. If $(X, \mathscr{F})$ is a foliated surface, a holomorphic curve C in X is *invariant* under $\mathscr{F}$ if the map $T_C \to T_X$ factors through $T_{\mathscr{F}}$.

Example 6. Consider the foliation of $\mathbb{A}^2$ generated by $xdy + \sqrt{2}ydx$, corresponding to $x\frac{\partial}{\partial x} - \sqrt{2}y\frac{\partial}{\partial y}$. The differential equations defining an invariant curve are

$$\dot{x}(t) = x(t), \quad \dot{y}(y) = -\sqrt{2}y(t).$$

There are two algebraic curves $\{x = 0\}$ and $\{y = 0\}$ which are invariant, and away from them the equations can be integrated to an analytic parametrization $x(t) = x_0e^t$, $y(t) = y_0e^{-\sqrt{2}t}$ defining local analytic curves $xy^{1/\sqrt{2}} = x_0y_0^{1/\sqrt{2}}$ which are *not* algebraic. (Note that if we replace $\sqrt{2}$ with a rational number, e.g. 2, then a multiple of the resulting 1-form has an algebraic integral; $x(xdy + 2ydx) = d(x^2y)$, and the leaves are algebraic; $x^2y = x_0^2y_0$.)

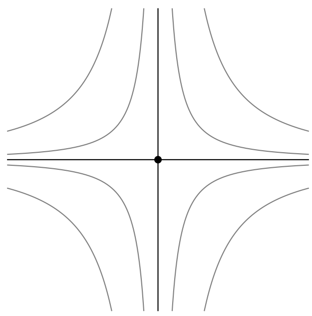

Figure 3: Leaves of $xdy + \sqrt{2}ydx$.

In fact, we have the following theorem (cf. [Voi, Thm 2.20]).

Theorem 1.5 (Frobenius)**.** *If X is a smooth complex analytic variety, and $\mathscr{F}$ is a foliation on X, then for any $x \in X \setminus \mathrm{Sing}(\mathscr{F})$ there is an analytic neighbourhood U of x and a holomorphic submersion $F : U \to W \subset \mathbb{C}^r$ such that $T_{\mathscr{F}}|_U = \ker(dF)$.*

Hence near smooth points we always have locally defined analytic leaves, given as the fibers of a holomorphic submersion. A key result for the local structure at singular points is the "separatrix theorem".

Definition 1.6. A *separatrix* at $p \in \mathrm{Sing}(\mathscr{F})$ is a local holomorphic curve C passing through p and invariant under $\mathscr{F}$.

Theorem 1.7 (Camacho–Sad, [CS82])**.** *If X is a smooth surface, with foliation $\mathscr{F}$, then through every $p \in \mathrm{Sing}(\mathscr{F})$ there exists at least one separatrix.*

In fact there are generalizations of this result to the case of singular X; see [Cam88].

Example 7. In Example 6 we found two separatrices: $\{x = 0\}$ and $\{y = 0\}$.

Remark. As a consequence of Theorem 1.5, if two invariant curves intersect (or if an invariant curve has a node), then the intersection point is a singularity of the foliation.

It is worth noting that while the leaves of the foliation can always be locally defined by an analytic equation, the global geometry can be quite pathalogical. Take the foliation from Example 6. It is hard to visualize the global geometry of the leaves for dimension reasons, but we can intersect with the torus $S^1 \times S^1 \subset \mathbb{C}^2$. The slices in $S^1 \times S^1$ of the leaves of $\mathscr{F}$ are parametrized as $x(t) = e^{it}$, $y(t) = e^{\phi - i\sqrt{2}t}$ for $t \in \mathbb{R}$, where $\phi \in \mathbb{R}$ is fixed for each leaf. We recover the classical example of a pathalogical immersion of $\mathbb{R}$ in the torus; each leaf is not just Zariski dense, but classically dense. This highlights one of the dangers of relying on pictures like Fig. 3 depicting only the real locus.

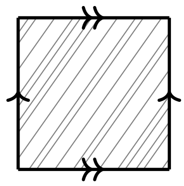

Figure 4: A slice of a single leaf of $xdy + \sqrt{2}ydx$.

1.3 Singularities

In view of Theorem 1.5 the local picture of $\mathscr{F}$ at smooth points is relatively simple, so we will be interested in the local theory of the singular points. We will look at some invariants of the singularities of $\mathscr{F}$ which can be used to identify classes of "nice" singularities out of all the possible modifications (under e.g. blowups) of a singularity, and can be accumulated to provide some global information about $\mathscr{F}$. We will only consider the case where the underlying surface is smooth.

1.3.1 Multiplicity

Fix a foliation $\mathscr{F}$ on a smooth surface X, with a singular point $p \in \mathrm{Sing}(\mathscr{F})$. Taking local analytic coordinates x, y near $p = (0, 0)$, we have a generator $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ for $\mathscr{F}$.

Definition 1.8. The *multiplicity* of the singularity p is

$$m(\mathscr{F}, p) \coloneqq \dim_{\mathbb{C}} \hat{\mathcal{O}}_{X,p}/(A, B).$$

We may then define a count of the total number of singularities of $\mathscr{F}$:

$$m(\mathscr{F}) \coloneqq \sum_{p \in \mathrm{Sing}(\mathscr{F})} m(\mathscr{F}, p),$$

which is a finite sum provided that X is compact.

Remark. One could also view $m(\mathscr{F}, p)$ as the multiplicity at p of the 0-dimensional subscheme $\mathrm{Sing}(\mathscr{F}) \subset X$ cut out by the ideal sheaf in Eq. (1).

Example 8. For the foliation given by $xdy + ydx$ a generating vector field is $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, so the multiplicity at the origin is

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/(x, y) = 1.$$

If we instead consider the foliation generated by $x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$, then the multiplicity at the origin is

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/(x, y^2) = 2.$$

Note that perturbing this to the foliation $x\frac{\partial}{\partial x} + (y^2 + \varepsilon)\frac{\partial}{\partial y}$ results in two singularities $(0, \pm\sqrt{\varepsilon})$ both with multiplicity 1.

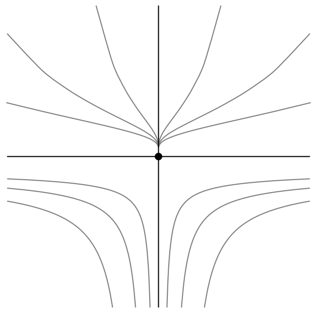

Figure 5: The foliation generated by $x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$.

Proposition 1.9. *Suppose $(X, \mathscr{F})$ is a compact foliated surface. Then*

$$m(\mathscr{F}) = T_{\mathscr{F}} \cdot T_{\mathscr{F}} + T_{\mathscr{F}} \cdot K_X + c_2(X) = c_2(X) - T_{\mathscr{F}} \cdot N_{\mathscr{F}}.$$

Proof. Note that the second equality is immediate from $K_{\mathscr{F}} = -T_{\mathscr{F}} = K_X + N_{\mathscr{F}}$, Proposition 1.2. For the first equality, note that $c_2(T_{\mathscr{F}}^\vee \otimes T_X)$ is given by the vanishing locus of the section in $H^0(T_{\mathscr{F}}^\vee \otimes T_X)$ corresponding to the inclusion $T_{\mathscr{F}} \to T_X$, which has isolated zeros at the singularities of $\mathscr{F}$. The intersection multiplicity with the zero section at $p \in \mathrm{Sing}(\mathscr{F})$ is precisely $m(\mathscr{F}, p)$, and so

$$m(\mathscr{F}) = c_2(T_{\mathscr{F}}^\vee \otimes T_X).$$

Now recall that

$$c_2(T_{\mathscr{F}}^\vee \otimes T_X) = c_2(T_X) + c_1(T_X)c_1(T_{\mathscr{F}}^\vee) + c_1(T_{\mathscr{F}}^\vee)^2,$$

which can be seen from the splitting principle:

$$\begin{aligned} c_2(L \otimes (L_1 \oplus L_2)) &= (c_1(L) + c_1(L_1))(c_1(L) + c_1(L_2)) \\ &= c_1(L_1)c_1(L_2) + (c_1(L_1) + c_1(L_2))c_1(L) + c_1(L)^2 \\ &= c_2(L_1 \oplus L_2) + c_1(L_1 \oplus L_2)c_1(L) + c_1(L)^2. \end{aligned}$$

The result then follows from $c_1(T_X) = -c_1(K_X)$, $c_1(T_{\mathscr{F}}^\vee) = -c_1(T_{\mathscr{F}})$. ☐

Example 9. Recall that the Chern character of $\mathbb{P}^n$ is $(1+h)^{n+1}$ for h a hyperplane class, so the Euler class is $(n+1)h^n$. Hence $c_2(\mathbb{P}^2) = 3$, so for a foliation of $\mathbb{P}^2$ we must have $m(\mathscr{F}) = 3 + T_{\mathscr{F}} \cdot T_{\mathscr{F}} + T_{\mathscr{F}} \cdot K_{\mathbb{P}^2}$. In Example 4 we found $T_{\mathscr{F}} = \mathcal{O}(1)$, so

$$m(\mathscr{F}) = 3 + H \cdot H + H \cdot (-3H) = 3 + 1 - 3 = 1.$$

Indeed, there was one singularity with the local form $y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, which has multiplicity 1. In fact the integer $T_{\mathscr{F}} \cdot T_{\mathscr{F}} + T_{\mathscr{F}} \cdot K_X$ must always be even, from computations with Stiefel–Whitney classes $w = \mathrm{Sq}(\nu)$ and their relation to the Wu and Chern classes (see [Mil], §8 and §14):

$$\begin{aligned} L \cdot K_X = -c_1(X)c_1(L) &\equiv w_2(X)w_2(L) && \text{reducing mod 2} \\ &\equiv (\nu_2(X) - w_1(X)^2)w_2(L) && \text{from } w = \mathrm{Sq}(\nu) \\ &\equiv \nu_2(X)w_2(L) && \text{odd } w_k \text{ vanish} \\ &\equiv w_2(L)^2 && \text{from } \mathrm{Sq}^k(\alpha_k) = \alpha_k^2 \\ &\equiv c_1(L)^2 \mod 2 \end{aligned}$$

for any line bundle L. Hence a foliation of $\mathbb{P}^2$ must have an odd number of singularities, and in particular at least one.

Remark. It can be shown directly that there is no exact sequence of the form $0 \to \mathcal{O}(a) \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}(b) \to 0$ on $\mathbb{P}^2$, so that there can be no smooth foliation, but this doesn't obviously generalize to a parity constraint.

1.3.2 Holonomy

Now we consider the case of a separatrix C through $p \in \mathrm{Sing}(\mathscr{F})$. A punctured neighbourhood of p in C is isomorphic to the punctured disc $\mathbb{D}^* \subset \mathbb{C}$, and has fundamental group generated by a loop γ. We may locally project the coordinates on X to C, and given a lift of the base of γ we obtain a unique path lifting γ in a neighbouring leaf of $\mathscr{F}$. This lifted path may no longer be a loop, and so taking the endpoint gives an endomorphism of the fiber of this projection over the basepoint. Writing D for this fiber, we have a homomorphism from $\pi_1(C \setminus \{p\}) \simeq \mathbb{Z}$ to (locally defined) biholomorphisms of D relative to p.

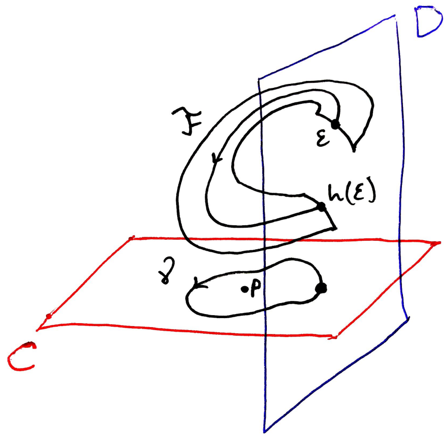

Figure 6: Holonomy of a foliation.

Definition 1.10. The *holonomy* of the separatrix C is the (locally defined) biholomorphism of a disc neighbourhood of p in D coming from a generator of $\pi_1(C \setminus \{p\})$ with positive orientation, which is defined up to conjugacy. In practice we will only concern ourselves with the derivative of this (locally defined) biholomorphism at p, which is then a well-defined element of $\mathbb{C}^*$.

To make things more concrete, suppose again that we have a local generator $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ for $\mathscr{F}$ near $p = (0,0)$, such that $C = \{y = 0\}$. Take γ to be the loop $t \mapsto (e^{2\pi i t}, 0)$ generating $\pi_1(C \setminus \{p\})$. We have the projection $(x, y) \mapsto (x, 0) \in C$, and for each $\varepsilon \in \mathbb{C}$ we lift γ to a path tangent to $\mathscr{F}$ starting at $(1, \varepsilon)$ which has an endpoint $(1, h(\varepsilon))$. The holonomy is the map $h(\varepsilon)$.

Example 10. Consider the foliation generated by $x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$. The leaves are locally fibers of the function $xy^{-\lambda}$, so lifting $(e^{2\pi i t}, 0)$ to a path in the leaf through $(1, \varepsilon)$ we obtain $(e^{2\pi i t}, \varepsilon e^{2\pi i \lambda t})$, with endpoint $(1, \varepsilon e^{2\pi i \lambda})$. Hence we get $h(\varepsilon) = \varepsilon e^{2\pi i \lambda}$, so the holonomy of the separatrix $\{y = 0\}$ is already linear with derivative $e^{2\pi i \lambda}$. Note that when $\lambda = -\sqrt{2}$ as in Example 6 this shows that the holonomy is an irrational rotation of infinite order, giving some evidence of the leaves not closing up nicely near the singularity.

Example 11. Consider the radial foliation generated by $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. The leaf through $(1, \varepsilon)$ is the line it spans through the origin, and so lifting the loop $(e^{2\pi i t}, 0)$ to this leaf gives $(e^{2\pi i t}, \varepsilon e^{2\pi i t})$, which is again a loop. Hence the holonomy is trivial in this example.

The holonomy of a separatrix actually determines the local properties of $\mathscr{F}$ at the singularity in p for certain classes of reduced singularities, due to results in [MM80], [MR82].

1.3.3 Eigenvalues

Near a singularity, as in the study of dynamical systems, we can consider the linearization of the generating vector field, replacing

$$A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$$

with

$$\left(\frac{\partial A}{\partial x}\bigg|_{(0,0)} \cdot x + \frac{\partial A}{\partial y}\bigg|_{(0,0)} \cdot y\right)\frac{\partial}{\partial x} + \left(\frac{\partial B}{\partial x}\bigg|_{(0,0)} \cdot x + \frac{\partial B}{\partial y}\bigg|_{(0,0)} \cdot y\right)\frac{\partial}{\partial y}.$$

The resulting linear system is integrable, with integral curves given by $t \mapsto e^{Jt}v_0$ for J the Jacobian matrix of coefficients, so the salient properties of the linearized dynamics are determined by the eigenvalues λ_1, λ_2 of the matrix J, which is determined (up to conjugation) by the choice of generating vector field.

Definition 1.11. If λ_1 and λ_2 are not both zero, we say that λ_1/λ_2 if $\lambda_2 \neq 0$, or λ_2/λ_1 if $\lambda_1 \neq 0$, is the *eigenvalue* of $\mathscr{F}$ at the singularity p, a complex number defined up to inversion. If the eigenvalue λ is not a positive rational number, we say that the singularity is *reduced.* If $\lambda \neq 0$ we say the reduced singularity is *non-degenerate*, otherwise it is a *saddle-node.*

Remark. Multiplication by a non-vanishing holomorphic function scales λ_1 and λ_2 proportionally, so that the eigenvalue is well-defined up to inversion in terms of $\mathscr{F}$ and p. The concept of a reduced singularity of a foliation is well-defined since the positive rational numbers are closed under inversion.

Example 12. The radial foliation generated by $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ (Fig. 2) has $\lambda = \lambda_1 = \lambda_2 = 1$, and so the origin is not a reduced singularity. On the other hand, the foliation generated by $x\frac{\partial}{\partial x} - \sqrt{2}y\frac{\partial}{\partial y}$ (Fig. 3) is reduced, with eigenvalue $-\sqrt{2}$. The singularity with positive multiplicity given by $x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$ (Fig. 5) is an example of a saddle-node.

In fact, reduced singularities have a classification with normal forms, depending on the values of λ (see [Br, §1]). A key property of this classification is that the separatrices at a reduced singularity form a local complete intersection, having at most 2 components. Compare this with the non-reduced radial foliation of Example 3, which has infinitely many separatrices. The significance of this class of singularities is that we can "reduce" to them after some sequence of blowups, Theorem 1.14, and moreover the blowup of a reduced singularity remains reduced.

1.4 Blowups

Example 13. Consider the foliation of $\mathbb{P}^2$ from Example 3. In coordinates $[1 : x : y]$ we have a singularity of the form $xdy - ydx$ with many separatrices, which we would like to blow up. Over this region, the blowup X is covered by charts $U, V \simeq \mathbb{A}^2$ where $U = \{(x, tx, 1 : t) \in \mathbb{A}^2 \times \mathbb{P}^1\}$, $V = \{(sy, y, s : 1) \in \mathbb{A}^2 \times \mathbb{P}^1\}$. The pullbacks to U and V of $\omega = zdy - ydz$ are

$$\omega|_U = xd(tx) - txdx = x^2dt, \quad \omega|_V = sydy - yd(sy) = -y^2ds.$$

These vanish to second order along the exceptional divisor E, but factoring this out we get 1-forms dt and $-ds$ on U and V which patch together to give a nowhere vanishing 1-form with values in the line bundle $\mathcal{O}(-2E)$. Pictorially this is exactly as expected.

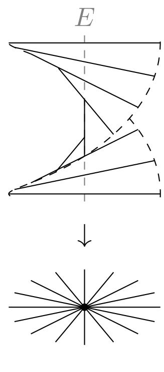

Figure 7: Blowing up $xdy - ydx$.

Example 14. If we instead have the singularity $\omega = xdy + ydx$, then

$$\omega|_U = x^2dt + 2xtdx, \quad \omega|_V = y^2ds + 2ysdy.$$

This has only first order vanishing along E, giving a 1-form valued in $\mathcal{O}(-E)$, but with two singularities from the local forms $xdt + 2tdx$, $yds + 2sdy$. The exceptional divisor takes the role of the other separatrix downstairs in lifting the singularity.

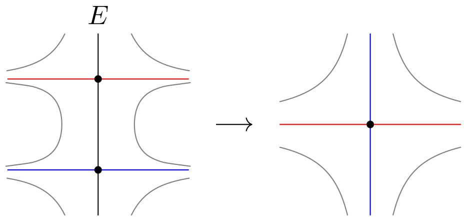

Figure 8: Blowing up $xdy + ydx$.

Example 15. Finally, consider blowing up a smooth foliation; $\omega = dx$. Then

$$\omega|_U = dx, \quad \omega|_V = sdy + yds,$$

so we have introduced a singularity on the exceptional divisor, which crosses the strict transform of the unique separatrix downstairs. Note that the pullback doesn't vanish along E.

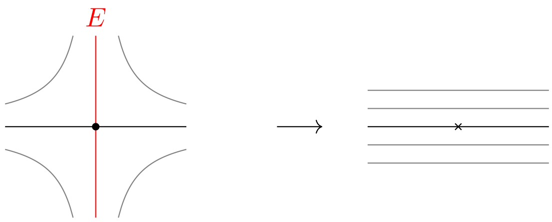

Figure 9: Blowing up a smooth foliation.

Definition 1.12. Summarizing what we have seen in these examples, if $(X, \mathscr{F})$ is a foliated surface and $p \in X$, then under the blowup $\pi : \tilde{X} = \mathrm{Bl}_p X \to X$ we have a rational section of $\pi^* N_{\mathscr{F}} \otimes \Omega^1_{\tilde{X}}$ with a zero of some order $a \geq 0$ along the exceptional divisor E. This then gives a regular section of $\pi^* N_{\mathscr{F}}(-aE) \otimes \Omega^1_{\tilde{X}}$, defining a foliation $\tilde{\mathscr{F}}$ on $\tilde{X}$ with $N_{\tilde{\mathscr{F}}} = \pi^* N_{\mathscr{F}}(-aE)$. We will refer to the vanishing order a as $a(\mathscr{F}, p)$.

Remark. By a similar argument, we can pullback the foliation $\mathscr{F}$ along any rational map due to the characterization of foliations in terms of rational sections of Ω^1_X, but in this generality we have less control over how the normal bundle changes.

In Example 15 we computed $a(\mathscr{F}, p) = 0$ for $p \notin \mathrm{Sing}(\mathscr{F})$, and in the other examples we found $a(\mathscr{F}, p) = 1$ for the reduced singularity $xdy + ydx$ and $a(\mathscr{F}, p) = 2$ for the non-reduced radial singularity. It is a fact that for reduced singularities $a(\mathscr{F}, p) \in \{0, 1\}$, which can be shown from the normal forms. Recall that $K_{\tilde{X}} = K_X + E$, so

$$K_{\tilde{\mathscr{F}}} = K_{\tilde{X}} + N_{\tilde{\mathscr{F}}} = \pi^* K_{\mathscr{F}} + (1 - a)E.$$

Hence for reduced singularities we get $K_{\tilde{\mathscr{F}}} \geq \pi^* K_{\mathscr{F}}$.

We now make a remark about the holonomy of the foliation $\tilde{\mathscr{F}}$. For each separatrix C through p, the strict transform $\tilde{C} \subset \tilde{X}$ is an invariant curve of $\tilde{\mathscr{F}}$ that meets the exceptional divisor E at a point q, which may or may not be a singularity of $\tilde{\mathscr{F}}$.

Proposition 1.13. *The holonomy of $\tilde{\mathscr{F}}$ along $\tilde{C}$ through q is the same as that of $\mathscr{F}$ along C through p.*

Proof. Indeed, the loop γ, curve D, and portions of leaves along which γ is lifted in the definition of the holonomy are all disjoint from p, and so lift isomorphically to the same constructions in $\tilde{X}$. □

Example 16. In Example 11 we saw that the radial fibration has trivial holonomy, which can be seen in light of this proposition from the fact that blowing it up gives a smooth foliation (Example 13).

We conclude the section by citing Seidenberg's theorem on reduced singularities.

Theorem 1.14 (Seidenberg, [Sei68]). *If $(X, \mathscr{F})$ is a foliated surface, and $p \in \mathrm{Sing}(\mathscr{F})$, then there is a sequence of blowups $\pi : \tilde{X} \to X$ of centres over p such that the induced foliation $\tilde{\mathscr{F}}$ has only reduced singularities on the fiber over p.*

The basic idea of the proof is to keep track of multiplicities when blowing up, allowing to pass to the case where the linearization is non-zero, and case analysis of the vanishing orders along the exceptional divisors allows to conclude. See [Br, §1] for details.

2 Foliations and curves

It is clear that a foliation interacts with curves on the surface, and in fact the intersection theory of the line bundles associated to the foliation can be computed in terms of local invariants. In this section we introduce two of these local invariants, and look at an example constraining a foliation using $K_{\mathscr{F}}$.

2.1 Tangency

For a curve which is not $\mathscr{F}$-invariant, the local leaves of $\mathscr{F}$ intersect it non-degenerately, and we can ask about the multiplicity of these intersections.

Definition 2.1. Suppose C is a curve, and $p \in C$. We define the *tangency order* of $\mathscr{F}$ to C at p as

$$\operatorname{tang}(\mathscr{F}, C, p) = \dim_{\mathbb{C}} \mathcal{O}_{X,p}/(f, v(f)),$$

where f is a local equation defining C and v is the vector field generating $\mathscr{F}$, viewed as a derivation. We may then define the total tangency order of $\mathscr{F}$ along C:

$$\operatorname{tang}(\mathscr{F}, C) = \sum_{p \in C} \operatorname{tang}(\mathscr{F}, C, p).$$

Remark. The component of C through p is $\mathscr{F}$-invariant iff $v(f)$ vanishes along it, i.e. is a multiple of f, and so $\operatorname{tang}(\mathscr{F}, C, p) < \infty$ iff this is not the case. If $p \notin \operatorname{Sing}(\mathscr{F}) \cup \operatorname{Sing}(C)$ then $\operatorname{tang}(\mathscr{F}, C, p) = 0$ if $\mathscr{F}$ and C are transverse at p, so the sum defining $\operatorname{tang}(\mathscr{F}, C)$ is a finite sum provided C has no $\mathscr{F}$-invariant components.

Proposition 2.2. *Suppose C has no $\mathscr{F}$-invariant components. Then*

$$T_{\mathscr{F}} \cdot C = C \cdot C - \operatorname{tang}(\mathscr{F}, C).$$

Proof. The local functions $v(f)$ glue to give a global section of $K_{\mathscr{F}} \otimes \mathcal{O}(C)$, since $\mathscr{F}$ is defined by a $T_{\mathscr{F}}^{\vee} = K_{\mathscr{F}}$-valued vector field and C is cut out by a $\mathcal{O}(-C)^{\vee} = \mathcal{O}(C)$-valued function. The tangency order $\operatorname{tang}(\mathscr{F}, C, p)$ is precisely the vanishing order of $v(f)|_C$ at p, as

$$\mathcal{O}_{X,p}/(f, v(f)) = \mathcal{O}_{C,p}/(v(f)|_C),$$

so

$$\operatorname{tang}(\mathscr{F}, C) = \deg(K_{\mathscr{F}} \otimes \mathcal{O}(C))|_C = K_{\mathscr{F}} \cdot C + C \cdot C.$$

□

Remark. Note that the proof actually gives an equivalence with the divisor $\sum_{p \in C} \operatorname{tang}(\mathscr{F}, C, p)p$, which is technically stronger than the equality of degrees if for example C is an elliptic curve.

Example 17. Consider the parallel foliation $\mathscr{F}$ on $\mathbb{P}^2$, generated by $\frac{\partial}{\partial x}$ on $\mathbb{A}^2$, and let C be the plane curve $y = x^d$. This is transverse to $\mathscr{F}$ except at the origin $p = [0:0:1]$ and the point $q = [0:1:0]$ at infinity. In coordinates $[x:1:z]$ the foliation is generated by $\frac{\partial}{\partial x}$ and C is given by $z^{d-1} = x^d$, so we get

$$\operatorname{tang}(\mathscr{F}, C, p) = \dim_{\mathbb{C}} \mathbb{C}[x, y]/(y - x^d, -dx^{d-1}) = d - 1,$$
$$\operatorname{tang}(\mathscr{F}, C, q) = \dim_{\mathbb{C}} \mathbb{C}[x, z]/(z^{d-1} - x^d, -dx^{d-1}) = (d - 1)^2.$$

On the other hand, from Example 4 we have $T_{\mathscr{F}} = \mathcal{O}(1)$, and $\mathcal{O}(C) = \mathcal{O}(d)$, so

$$C \cdot C - T_{\mathscr{F}} \cdot C = d^2 - d = (d - 1) + (d - 1)^2$$

as expected.

2.2 Vanishing order

For a curve C which is $\mathscr{F}$-invariant, the 1-form defining $\mathscr{F}$ vanishes along the curve, and we can ask about the order of this vanishing. Suppose f is a local equation defining C, and ω is the 1-form defining $\mathscr{F}$ near p. Since ω vanishes on C, we can write $g \cdot \omega = h \cdot df + f \cdot \eta$ for some holomorphic functions g, h and 1-form η, where h may be assumed coprime to f, [Lin86]. (Roughly speaking this comes from $\Omega^1_C = \mathcal{O}_X/(f) \otimes \Omega^1_X/(df)$.) The meromorphic function $(h/g)|_C$ is in fact independent of this decomposition, being the residue of $\frac{\omega}{f} = \frac{h}{g}\frac{df}{f} + \frac{\eta}{g}$ along C.

Definition 2.3. We define the *vanishing index* of $\mathscr{F}$ along C at p as

$$Z(\mathscr{F}, C, p) = \operatorname{ord}_p \left(\frac{h}{g}\right)\bigg|_C,$$

which is independent of the choice of defining equation f and 1-form ω since multiplication by a non-vanishing holomorphic function does not affect vanishing order. We may then define the total vanishing index of $\mathscr{F}$ along C:

$$Z(\mathscr{F}, C) = \sum_{p \in C} Z(\mathscr{F}, C, p).$$

Remark. If $p \notin \operatorname{Sing}(\mathscr{F})$, then by Theorem 1.5 we may assume $\omega = df$, so $h/g = 1$ and $Z(\mathscr{F}, C, p) = 0$. So really this index is an invariant of separatrices. This shows that the sum defining $Z(\mathscr{F}, C)$ is finite.

Remark. The meromorphic function h/g may actually have a pole at p, so that $Z(\mathscr{F}, C, p)$ is negative. Consider the foliation given by $\omega = 7ydx - 4xdy$, with separatrix cut out by $f = x^7 - y^4$. Then

$$\omega = \frac{xy \cdot df - 4xdy \cdot f}{x^4},$$

so the index is the order of vanishing of y/x^{4-1} on $C = \{f = 0\}$ at $p = (0,0)$. From the normalization $t \mapsto (t^4, t^7)$ we compute $Z(\mathscr{F}, C, p) = 7 - 4(4-1) = -5$. Fortunately this only occurs for singularities with infinitely many separatrices (necessarily non-reduced), and when C is smooth we get agreement with the Poincaré–Hopf index at p of the vector field generating the foliation; [Bru97].

Proposition 2.4. *Suppose C is an $\mathscr{F}$-invariant curve. Then*

$$N_{\mathscr{F}} \cdot C = C \cdot C + Z(\mathscr{F}, C).$$

Proof. The local 1-forms ω/f are globally valued in $N_{\mathscr{F}} \otimes \mathcal{O}(-C)$, and by taking residues along C the local functions $(h/g)|_C$ glue to give a meromorphic section of $(N_{\mathscr{F}} \otimes \mathcal{O}(-C))|_C$. By definition we then have

$$Z(\mathscr{F}, C) = \deg(N_{\mathscr{F}} \otimes \mathcal{O}(-C))|_C = N_{\mathscr{F}} \cdot C - C \cdot C.$$

□

Remark. As above, this proof actually gives an equivalence with the divisor $\sum_{p \in C} Z(\mathscr{F}, C, p)p$.

2.3 A rational example

Suppose $\mathscr{F}$ is a foliation of $\mathbb{P}^2$, and suppose the tangency order $\operatorname{tang}(\mathscr{F}, \ell)$ is d for a general line ℓ. Then Proposition 2.2 gives

$$K_{\mathscr{F}} \cdot \ell = d - \ell \cdot \ell = d - 1, \qquad (*)$$

so $K_{\mathscr{F}} = \mathcal{O}(d-1)$. There is a natural tricohotomy between the case $d = 0$, where $K_{\mathscr{F}}$ is negative, the case $d = 1$, where $K_{\mathscr{F}}$ is trivial, and the general case $d \geq 2$. We will consider the case $d = 0$.

Specifically, we are assuming that the general line ℓ is transverse to $\mathscr{F}$. In fact, by the formula $(*)$ applying Proposition 2.2 again we have $\operatorname{tang}(\mathscr{F}, \ell) = 0$ for *any* non-$\mathscr{F}$-invariant line ℓ, so all non-$\mathscr{F}$-invariant lines are transverse to $\mathscr{F}$. Now at any point in $\mathbb{P}^2$ which is not a singularity of $\mathscr{F}$, there is a line through the point sharing the tangent direction of $\mathscr{F}$, which is by construction not transverse to $\mathscr{F}$. This line must be $\mathscr{F}$-invariant, and so we see that $\mathscr{F}$ is a pencil of lines in $\mathbb{P}^2$.

So we have shown, a foliation of $\mathbb{P}^2$ with $K_{\mathscr{F}}$ negative must be a pencil of lines (as in Example 3).

3 Fibrations

In this section we look at a few examples of foliations arising from fibrations.

3.1 Algebraic integrability

It is a natural question to ask when the leaves of a foliation are algebraic curves. Examples we have seen are $xdy - ydx$, with leaves $ax + by = 0$ for $[a : b] \in \mathbb{P}^1$, and $pxdy + qydx$ for $p, q \in \mathbb{N}$, with leaves $x^q y^p = c$ for $c \in \mathbb{A}^1$. Notice that $xdy - ydx = y \cdot d(x/y)$, and $pxdy + qydx = d(x^q y^p)/(x^{q-1} y^{p-1})$, so these are examples of the following construction.

Definition 3.1. Given a rational map $f : X \dashrightarrow C$, where X is a surface and C is a curve, the relative tangent bundle $T_{X/C}$ defines a foliation on X which we call a *fibration*. The leaves of this fibration are precisely the fibers of f, which are algebraic curves.

In the converse direction, there is a general statement which can be made:

Proposition 3.2. *Suppose $(X, \mathscr{F})$ is a foliated surface, such that for a general $x \in X$ the leaf of $\mathscr{F}$ through x is algebraic. Then there exists a rational map $X \dashrightarrow Y$ such that $T_{X/Y}$ and $T_{\mathscr{F}}$ are isomorphic on a dense open subset of X.*

Sketch proof. We consider the Hilbert scheme Hilb_X. Tangency to the foliation $\mathscr{F}$ is a condition on subvarieties which cuts out a closed subscheme $\mathcal{T}_{X,\mathscr{F}} \subseteq \mathrm{Hilb}_X$. By assumption, the canonical map from the universal family $\mathcal{U}$ over $\mathcal{T}_{X,\mathscr{F}}$ to X is dominant. Since $\mathcal{U}$ is an infinite disjoint union of projective components, by the Baire category theorem we have some such component $Z \subset \mathcal{U}$ over $Y \subset \mathcal{T}_{X,\mathscr{F}}$ which maps surjectively to X. After restricting to a suitable hyperplane Y^0 in Y, we get a finite surjection $Z^0 \to X$, which lifts to a Galois covering $\tilde{Z}^0 \to X$, where the Galois group G can be made to act on a base $\tilde{Y}^0$.

$$\begin{tikzcd} \tilde{Z}^0 \arrow[r] \arrow[d] \arrow[rrr, bend left, "\text{Galois}"] & Z^0 \arrow[r, hook] \arrow[d] & \mathcal{U} \arrow[r] \arrow[d] & X \\ \tilde{Y}^0 \arrow[r] & Y^0 \arrow[r, hook] & \mathcal{T}_X & \end{tikzcd}$$

At this point we can collapse the fibers by taking the quotient $\tilde{Z}^0/G$, which maps birationally to X, and so we get a rational map $X \dashrightarrow \tilde{Y}^0/G$ which by construction has fibers tangent to $\mathscr{F}$. □

3.2 Miyaoka's criterion

We saw in Section 2.3 an example where $K_{\mathscr{F}}$ being negative forces the foliation $\mathscr{F}$ to consist of rational curves. We present a theorem due to Miyaoka which asserts a more general result of this form.

Theorem 3.3 (Miyaoka)**.** *Suppose $(X, \mathscr{F})$ is a foliated surface, and H is an ample divisor on X such that $K_{\mathscr{F}} \cdot H < 0$. Then there is a birational morphism $\tilde{X} \to X$ and a fibration $f : \tilde{X} \to C$ such that the general fiber of f is rational, and $\mathscr{F}$ is the foliation induced by f.*

Proof (Bogomolov, McQuillan). By Bertini's theorem, for $n \gg 0$ we have some smooth curve $C \in |nH|$ which is disjoint from $\mathrm{Sing}(\mathscr{F})$. Write $Y = X \times C$, and consider the rank 1 foliation $\mathscr{G}$ on Y given by $\mathscr{F}$ on each fiber. The diagonal $D \subset C \times C \subset Y$ is disjoint from $\mathrm{Sing}(\mathscr{G})$, and integrating $\mathscr{G}$ at the points of D gives a local analytic surface V through D.

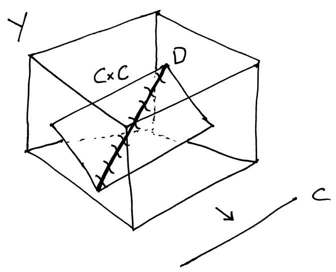

Figure 10: The foliation $\mathscr{G}$ near D.

By construction $N_{D/V} = T_{\mathscr{G}}|_D$, which is identified with $T_{\mathscr{F}}|_C$, so $\deg N_{D/V} = T_{\mathscr{F}} \cdot C = nT_{\mathscr{F}} \cdot H > 0$. Hence $N_{D/V}$ is ample, and this forces the surface V to be algebraic by a theorem of Andreotti; see [Bos13, Thm 3.4]. Write S for its Zariski closure. We have shown that the leaves of $\mathscr{G}$ through D are algebraic, and hence the leaves of $\mathscr{F}$ are algebraic. By Proposition 3.2 this means that $\mathscr{F}$ is induced by a fibration, and it suffices to show that the general fiber is rational.

We have a fibration $f : S \to C$, inducing a foliation $\mathscr{H}$ on S which agrees with $\mathscr{G}$ locally around D. The fibers of f are reduced, so $K_{\mathscr{H}} = K_{S/C}$. Now by Fujita semipositivity, if the general fiber is not rational then $K_{S/C}$ is a sum of a nef divisor and an effective disivor, and in particular $K_{S/C}$ is nef. But $K_{S/C} \cdot D = K_{\mathscr{H}} \cdot D = K_{\mathscr{F}} \cdot C < 0$, so the general fiber must be rational. Since the fibers of f correspond to the leaves of $\mathscr{F}$, we are done. □

Remark. The condition in this theorem is necessary; suppose $f : X \to C$ is a fibration inducing a foliation $\mathscr{F}$ whose general fiber is rational. Then $K_{\mathscr{F}} = K_{X/C} - D$ where $D \geq 0$ is supported on fibers of f. If F is a general fiber, then $K_{\mathscr{F}} \cdot F = K_X \cdot F = -2$ by adjunction. Moreover F is nef, so if H is ample we have that $nF + H$ is also ample for $n \geq 0$, while $K_{\mathscr{F}} \cdot (nF + H) = -2n + K_{\mathscr{F}} \cdot H$ is negative for $n \gg 0$.

4 The Green–Griffiths conjecture and stability

We introduce the Green–Griffiths conjecture, and sketch some of the ideas due to Bogomolov and McQuillan applying the theory of foliations to the problem. Motivated by an example of these ideas, we digress into a discussion of some basic concepts of slope stability for sheaves.

4.1 The Green–Griffiths conjecture

The Green–Griffiths conjecture concerns the transcendental geometry of general type varieties.

Conjecture (Green–Griffiths, [GG80])**.** *Let X be a smooth complex projective variety of general type. Then X admits no Zariski dense entire holomorphic curves.*

There is a more general form of the conjecture, as follows:

Conjecture (Green–Griffiths–Lang, [Lan86])**.** *Let X be a smooth complex projective variety of general type. There is a proper algebraic subvariety $Y \subset X$ such that every entire holomorphic curve $f : \mathbb{C} \to X$ factors through Y.*

We are considering a holomorphic map $f : \mathbb{C} \to X$, which induces also a map $df : \mathbb{C} \to \mathbb{P}(T_X)$. Writing $\mathcal{O}(-1)$ for the relative tautological bundle on $\mathbb{P}(T_X)$, we have (tautologically) $(df)^*\mathcal{O}(-1) = T_C$. Now suppose $\mathcal{O}(1)$ is big. This assumption is intended as an approximation of the condition of being general type. We may find an effective divisor $D \geq 0$ in $\mathbb{P}(T_X)$ representing $\mathcal{O}(1)$, which is locally a section of the bundle over X, although possibly not globally. Up to finite degree issues, this gives a rational section of $\mathbb{P}(T_X)$, which as mentioned earlier is precisely the data of a foliation on X, call it $\mathscr{F}$.

Now from $(df)^*\mathcal{O}(-1) = T_C$ we get $(df)^*\mathcal{O}(1) = K_C = 2g(C) - 2$, so if C is rational $C \cdot D < 0$. But this forces C to be a component of D, and so in fact $\mathscr{F}$ is tangent to C. In other words, from the assumption that $\mathcal{O}(1)$ is big we have produced a foliation on X which is tangent to every rational curve in X. The existence of such a foliation is a strong constraint, and this argument is part of work due to Bogomolov and McQuillan [M98] which lead to certain restricted results related to the conjecture.

4.2 Ample vector bundles

We would like to better understand this condition of $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ being big in terms of X and T_X directly. It turns out to be equivalent to a notion of "ample" for the vector bundle T_X.

Proposition 4.1 ([Har66], 3.2)**.** *Let $\mathcal{E}$ be a vector bundle on a variety X. Then $\mathcal{E}$ is an ample vector bundle on X iff the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$ on $\mathbb{P}(\mathcal{E}^\vee)$ is ample.*

We recall from [Har66] the notion of an ample vector bundle:

Definition 4.2. We say a vector bundle $\mathcal{E}$ over a variety X is *ample* if for any coherent sheaf $\mathcal{F}$ the sheaf $\mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E}$ is generated by global sections for $n \gg 0$.

Remark. When $\mathcal{E} = \mathcal{L}$ is a line bundle we have $\operatorname{Sym}^n \mathcal{L} = \mathcal{L}^{\otimes n}$, so this recovers the definition of ample for line bundles.

Remark. Motivated by the analogous statement for line bundles, one might ask whether this definition is equivalent to requiring $\operatorname{Sym}^n \mathcal{E}$ to give an embedding into the corresponding Grassmannian for $n \gg 0$. In fact this is not true: take for example $\mathcal{O} \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$. This is not ample, since $\mathcal{O}(-1) \otimes \operatorname{Sym}^n(\mathcal{O} \oplus \mathcal{O}(1))$ has a summand of $\mathcal{O}(-1)$ for all $n \geq 0$. However, the complete system $(1 \oplus 0, 0 \oplus x, 0 \oplus y)$ induces an embedding $\mathbb{P}^1 \hookrightarrow \operatorname{Gr}(2, \mathbb{C}^3)$; $[x : y] \mapsto \langle (1,0,0), (0,x,y) \rangle$.

The key point to proving Proposition 4.1 is the following fact:

Lemma 4.3. *If $p : \mathbb{P}(\mathcal{E}^\vee) \to X$ is the projection map, then for every coherent sheaf $\mathcal{F}$ on X we have*

$$\mathbf{R}p_*(p^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(n)) = \mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{E}).$$

Proof. By the projection formula we have

$$\mathbf{R}p_*(p^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(n)) = \mathcal{F} \otimes^{\mathbf{L}} \mathbf{R}p_*\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(n),$$

and $\mathbf{R}p_*\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(n) = \operatorname{Sym}^n \mathcal{E}$ as a relative version of

$$H^k(\mathcal{O}_{\mathbb{P}(V)}(n)) = \begin{cases} \operatorname{Sym}^n V^\vee & k = 0,\ n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

Motivated by the construction in the previous section, suppose we want to construct a general type surface X such that $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ is big. From Proposition 4.1, this is equivalent to requiring that T_X is ample. We will consider a degree d hypersurface in $\mathbb{P}^3$. There is the normal bundle exact sequence

$$0 \to N^\vee_{X/\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}|_X \to \Omega^1_X \to 0,$$

and $N_{X/\mathbb{P}^3} = \mathcal{O}(-d)$ by adjunction. Taking determinants we see that $\mathcal{O}(-4) = K_{\mathbb{P}^3} = K_X \otimes \mathcal{O}(-d)$, so at least $\det \Omega^1_X = K_X$ is ample for $d \gg 0$. It is not true that a vector bundle with ample determinant bundle is always itself ample (although the converse holds). However, in this particular case we can try to apply the following result:

Theorem 4.4 ([FL22], 3.10)**.** *If $\mathcal{E}$ is a slope semistable vector bundle, and the discriminant*

$$\Delta(\mathcal{E}) = 2 \operatorname{rank} \mathcal{E} \cdot c_2(\mathcal{E}) - (\operatorname{rank} \mathcal{E} - 1) c_1(\mathcal{E})^2$$

is zero, then $\mathcal{E}$ is ample iff $\det \mathcal{E}$ is ample.

We will see in the next section that Ω^1_X is slope semistable (in fact slope stable), after first introducing the notion of slope stability. However, in computing the discriminant we find

$$\begin{aligned} \Delta(\Omega^1_X) &= 4c_2(X) - c_1(X)^2 \\ &= 4\chi(X) - K_X \cdot K_X \\ &= 4\chi(X) - d(d-4)^2. \end{aligned}$$

In fact $\chi(X) = d(d^2 - 4d + 6)$ [Max24], giving $\Delta(\Omega^1_X) = 3d^2 - 8d + 8$. This is positive for all d, and so unfortunately the theorem doesn't apply in our example.

Example 18. For an example where $\det \mathcal{E}$ is ample but $\mathcal{E}$ is *not* ample, consider $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(2)$ on $\mathbb{P}^1$. Then $\det \mathcal{E} = \mathcal{O}(1)$ is clearly ample, but $\mathcal{E}$ cannot be ample as the summand $\mathcal{O}(-1)$ is not ample.

Remark. To see that $\det \mathcal{E}$ is ample whenever $\mathcal{E}$ is ample, note that $\operatorname{Sym}^n \mathcal{E}$ is ample and generated by global sections for $n \gg 0$. Tensoring with a globally generated sheaf preserves ampleness, so $(\operatorname{Sym}^n \mathcal{E})^{\otimes r}$ is ample, and hence the quotient $\wedge^r \operatorname{Sym}^n \mathcal{E} = \det(\operatorname{Sym}^n \mathcal{E})$ is also ample. But this line bundle is acted on by $\operatorname{GL}(\mathcal{E})$, and the only characters of GL are powers of the determinant, so $\det(\operatorname{Sym}^n \mathcal{E}) = (\det \mathcal{E})^{\otimes m}$ for some m. Hence $\det \mathcal{E}$ is ample.

4.3 Slope stability

The notion of slope stability is motivated by the application of Mumford's geometric invariant theory in the construction of moduli spaces of sheaves; it gives restricted classes of sheaves for which we can construct well-behaved moduli spaces via Grothendieck's Quot construction.

Definition 4.5. Suppose X is a complex projective variety, with ample line bundle $\mathcal{O}(1)$. For a torsion-free coherent sheaf $\mathcal{E}$ on X, we define the *slope* of $\mathcal{E}$ to be

$$\mu(\mathcal{E}) = \deg \mathcal{E} / \operatorname{rank} \mathcal{E},$$

where $\deg \mathcal{E}/(\dim X)!$ is the coefficient of $m^{\dim X}$ in the Hilbert polynomial $m \mapsto \chi(\mathcal{E} \otimes \mathcal{O}(m))$. We say $\mathcal{E}$ is slope *semistable* if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$, and *stable* if the inequality is strict. A direct sum of stable sheaves of the same slope is called *polystable*.

We will only be considering slope stability, and so we drop the slope prefix and write simply "$\mathcal{E}$ is (semi/poly)stable".

Remark. Line bundles are trivially stable, having no subsheaves of smaller rank.

4.3.1 Relation to ample vector bundles

To apply Proposition 4.1 in the situation of Ω^1_X above, we appeal to two major theorems:

Theorem 4.6 (Aubin [Aub82], Yau [Yau78])**.** *For a general type smooth complex projective variety, the tangent bundle admits a Kähler–Einstein metric.*

Theorem 4.7 ([Kob87], [Lüb83])**.** *If the tangent bundle of a smooth complex projective variety admits a Kähler–Einstein metric, then it is stable.*

The stability of a vector bundle is equivalent to stability of its dual (Proposition 4.11), so these results imply stability of Ω^1_X in our example earlier.

4.3.2 The Donaldson–Uhlenbeck–Yau Theorem

A key theorem in the theory of slope stability is the following theorem, formerly known as the Calabi conjecture.

Theorem 4.8 (Donaldson–Uhlenbeck–Yau)**.** *If X is a smooth complex projective manifold, then a holomorphic vector bundle $\mathcal{E}$ on X is polystable iff it admits a Hermite–Einstein metric.*

Definition 4.9. On a compact Kähler manifold (X, ω) with a holomorphic vector bundle $\mathcal{E}$, a Hermitian metric h on $\mathcal{E}$ is *Hermite–Einstein* if its Chern connection A has curvature F_A satisfying

$$F_A \wedge \omega^{\dim X - 1} = \lambda \cdot (\mathrm{id}_{\mathcal{E}} \otimes \omega^{\dim X})$$

for some $\lambda \in \mathbb{C}$.

In fact the constant λ is geometrically constrained:

$$\begin{aligned} \deg \mathcal{E} &= \int_X c_1(\mathcal{E}) \wedge \omega^{\dim X - 1} \\ &= \frac{i}{2\pi} \int_X \operatorname{tr}(F_A) \wedge \omega^{\dim X - 1} \\ &= \frac{i}{2\pi} \lambda \operatorname{rank} \mathcal{E} \operatorname{vol}(X), \end{aligned}$$

so $\lambda = -2\pi i \mu(\mathcal{E}) / \operatorname{vol}(X)$, which indicates how slope stability might be related. Indeed, the existence of a Hermite–Einstein metric given polystability is the hard part of the theorem; the other direction is relatively easy.

We now turn to look at some basic properties of stability.

Proposition 4.10. *A torsion-free coherent sheaf $\mathcal{E}$ is stable iff $\mathcal{E} \otimes \mathcal{L}$ is stable for any line bundle $\mathcal{L}$.*

Proof. Any subsheaf of $\mathcal{E}\otimes\mathcal{L}$ is of the form $\mathcal{F}\otimes\mathcal{L}$ for a subsheaf $\mathcal{F}$ of $\mathcal{E}$ since $-\otimes\mathcal{L}$ is an exact equivalence, so this follows from the formula

$$\mu(\mathcal{F}\otimes\mathcal{L}) = \mu(\mathcal{F}) + \deg(\mathcal{L}).$$

□

Proposition 4.11. *A vector bundle $\mathcal{E}$ is stable iff $\mathcal{E}^\vee$ is stable.*

Proof. Since $\mathcal{E} = \mathcal{E}^{\vee\vee}$, it suffices to prove the forward implication. Now $\mathcal{E}^\vee = \mathcal{E}\otimes(\det\mathcal{E})^\vee$, since a volume form defines a perfect pairing $\mathcal{E}\otimes\mathcal{E}\to\mathbb{C}$, so the result follows from Proposition 4.10. □

Proposition 4.12. *If*

$$0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$$

is a short exact sequence of torsion-free coherent sheaves, then

$$\min\{\mu(\mathcal{E}'),\mu(\mathcal{E}'')\} \le \mu(\mathcal{E}) \le \max\{\mu(\mathcal{E}'),\mu(\mathcal{E}'')\},$$

with equality at either end iff $\mu(\mathcal{E}') = \mu(\mathcal{E}) = \mu(\mathcal{E}'')$ or one of $\mathcal{E}'$ and $\mathcal{E}''$ vanishes.

Proof. Recall that degree and rank are additive in exact sequences. Hence

$$\mu(\mathcal{E}) = \frac{\deg\mathcal{E}' + \deg\mathcal{E}''}{\operatorname{rank}\mathcal{E}' + \operatorname{rank}\mathcal{E}''} = \frac{\operatorname{rank}\mathcal{E}'}{\operatorname{rank}\mathcal{E}' + \operatorname{rank}\mathcal{E}''}\cdot\mu(\mathcal{E}') + \frac{\operatorname{rank}\mathcal{E}''}{\operatorname{rank}\mathcal{E}' + \operatorname{rank}\mathcal{E}''}\cdot\mu(\mathcal{E}''),$$

which is a positive weighted average of total weight 1. If the endpoints of the average are distinct, we can only have an equality if one of the weights vanishes. □

Proposition 4.13. *A torsion-free coherent sheaf $\mathcal{E}$ is semistable (resp. stable) iff for all quotients $\mathcal{G}$ of $\mathcal{E}$ with $0 < \operatorname{rank}\mathcal{G} < \operatorname{rank}\mathcal{E}$ we have $\mu(\mathcal{G}) \ge \mu(\mathcal{E})$ (resp. $\mu(\mathcal{G}) > \mu(\mathcal{E})$).*

Proof. This is immediate from Proposition 4.12. □

Proposition 4.14. *If*

$$0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$$

is a short exact sequence of nonzero torsion-free coherent sheaves with $\mu(\mathcal{E}') = \mu(\mathcal{E}) = \mu(\mathcal{E}'')$, then $\mathcal{E}$ is semistable iff $\mathcal{E}'$ and $\mathcal{E}''$ are both semistable.

Proof. If $\mathcal{E}$ is semistable then $\mathcal{E}'$ is semistable by transitivity of subsheaves, and similarly for $\mathcal{E}''$ using Proposition 4.13. For the converse, assume $\mathcal{E}'$ and $\mathcal{E}''$ are semistable, and $\mathcal{F}$ is a subsheaf of $\mathcal{E}$. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0, \end{array}$$

where $\mathcal{F}' = \mathcal{E}'\cap\mathcal{F}$. If $\mathcal{F}' = 0$ then $\mu(\mathcal{F}) = \mu(\mathcal{F}'') \le \mu(\mathcal{E}'') = \mu(\mathcal{E})$, and similarly for $\mathcal{F}'' = 0$. Otherwise $\mu(\mathcal{F}') \le \mu(\mathcal{E}') = \mu(\mathcal{E})$, and similarly for $\mathcal{F}''$, so $\mu(\mathcal{F}) \le \mu(\mathcal{E})$ by Proposition 4.12. □

Corollary 4.15. *If $\mathcal{E},\mathcal{E}'$ are semistable with the same slope, then $\mathcal{E}\oplus\mathcal{E}'$ is also semistable with that same slope. Hence polystable implies semistable.*

Remark. If $\mu(\mathcal{E}) \ne \mu(\mathcal{E}')$, then $\mathcal{E}\oplus\mathcal{E}'$ is not semistable in general. For example, consider $\mathcal{O}\oplus\mathcal{O}(1)$ on $\mathbb{P}^1$. We have $\mu(\mathcal{O}\oplus\mathcal{O}(1)) = 1/2$, while $\mu(\mathcal{O}(1)) = 1$.

We conclude by noting the following innocuous-looking fact.

Theorem 4.16 (Narasimhan–Sedhadri [NS65])**.** *If $\mathcal{E},\mathcal{E}'$ are semistable vector bundles, then so is $\mathcal{E}\otimes\mathcal{E}'$.*

This fact can be seen from the Donaldson–Uhlenbeck–Yau Theorem, because tensor products of Hermite–Einstein metrics are again Hermite–Einstein; for $h = h_1\otimes h_2$ we have $F_A = F_{A_1}\otimes 1 + 1\otimes F_{A_2}$. For a long time there have been no elementary approaches found for proving this theorem, and it is worth noting that the statement fails in positive characteristic even for curves. See [Mac18] for a nice exposition on the context of this tensor product theorem, and the relation to representation theory.

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