Matrix Factorizations and Knörrer Periodicity

Calum Crossley; DOGS, 14th Jan 2025

Overview

- Introduction to matrix factorizations
- Motivation for Knörrer periodicity
- Examples with flips / flops

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- Motivation for Knörrer periodicity
- Examples with flips / flops
- (+ hopefully more contextual comments as we go)

Matrix factorizations in the wild

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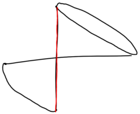

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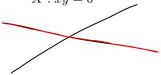
$X : xy = 0$



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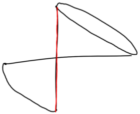

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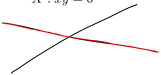

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(Eisenbud '80) Hypersurface singularities: eventually 2-periodic free resolutions, by matrices factoring the equation.

- Stable part: matrix factorization ($\leftrightarrow$ maximal Cohen-Macaulay module)
- Leftover: bounded complex of frees, i.e. $\mathfrak{Perf}$

Domesticated matrix factorizations

For a function W on a variety X, we want to define a ('derived' dg-/triang.) category $\mathrm{MF}(X, W)$ of matrix factorizations, with $\mathrm{MF}(X, 0) = D^b(X)$.

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Gives a $\mathbb{Z}/2$-differential graded category, for 'derived' homotopy category use injective modules, or define quasi-isomorphisms (subtle). Gives $\mathsf{MF}(X, W)$ but only $\mathbb{Z}/2$-graded.

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[Segal, "Equivalences between GIT quotients..."]

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- Forgetful functor to $\mathbb{Z}/2$-graded version from ± 1-eigendecomposition.
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Grading is a headache!

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Derived functors (with usual boundedness caveats):

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an object in $\mathrm{MF}(\mathbb{A}^2, xy - xy) = D^{b,\mathbb{Z}/2}(\mathbb{A}^2)$ quasi-isomorphic to $\mathcal{O}/(x, y)$, so $\mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathbf{R}\Gamma(\mathcal{O}/(x, y)) = \mathbb{C}$.

Relation to singularities

$$MF(X,W) \longleftrightarrow \mathrm{Sing}(\{W=0\})$$

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$$\mathcal{E}^0 \rightleftarrows \mathcal{E}^1 \quad \longleftrightarrow \quad \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^0 \to$$

Recall for hypersurface singularities: eventually 2-periodic resolutions; an MF then a bounded complex.

For smooth X, the $\mathbb{Z}/2$-graded MF(X, W) is equivalent to Orlov's derived category of singularities $D_{\mathrm{Sg}}(\{W = 0\}) := D^b(\{W = 0\})/\mathfrak{Perf}(\{W = 0\})$.

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MF(X, W) is "like" $D^b(\mathrm{Crit}(W))$, but this is only true on the nose for quadratic order singularities.

Classical Knörrer periodicity

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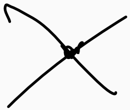

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$$\text{Koszul: } \left(\mathcal{O} \underset{\frac{1}{2}(x_1+x_2, -(y_1+y_2))}{\overset{(y_1-y_2, x_2-x_1)^T}{\rightleftarrows}} \mathcal{O}^2 \underset{\frac{1}{2}(y_1+y_2, x_1+x_2)^T}{\overset{(x_1-x_2, y_1-y_2)}{\rightleftarrows}} \mathcal{O} \right) \overset{\text{q.i.}}{\simeq} \mathcal{O}_\Delta.$$

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These are isomorphic by a linear change of coordinates. □

K.P. Shipman

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Classical K.P. $\mathsf{MF}(X \times \mathbb{A}^2, W + xy) \simeq \mathsf{MF}(X, W)$. (Knörrer '87)

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He was studying $f(x, y) + z_1^2 + \cdots + z_n^2$, found 2-periodicity (above), and almost 1-periodicity:

$$\mathsf{MF}(\mathbb{A}^{n+1}, F + z^2) \to \mathsf{MF}(\mathbb{A}^n \times [\mathbb{A}^1/\{\pm 1\}], F + z^2) \simeq \mathsf{MF}(\mathbb{A}^n, F).$$

Modern Knörrer periodicity

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So to apply Knörrer periodicity in a family, we want:

- plane bundle with a non-degenerate quadratic form, *and*
- global choice of one isotropic subspace. $\to$ line bundle summand $\mathscr{L}$.

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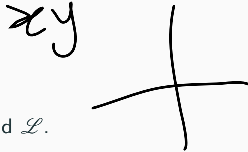

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[Shipman, "A geometric approach..."]

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Example: For a complete intersection $X = \{f_1 = \cdots = f_n = 0\}$, we get

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_n), f_1 p_1 + \cdots + f_n p_n).$$

Examples

$f \in H^0(\mathcal{O}(d))$

Degree d hypersurface $X = \{f(x) = 0\} \subset \mathbb{P}^n$:

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Elliptic curve $E \subset \mathbb{P}^2$ through a point $(0:0:1)$. Then $f = xP + yQ$, and
$\begin{pmatrix} x & Q \\ -y & P \end{pmatrix} \cdot \begin{pmatrix} P & -Q \\ y & x \end{pmatrix}$ in $\mathrm{MF}([\mathbb{A}^3/\mu_3], f)$ corresponds to the point $(0:0:1)$.

Examples

Nodal curve: $Y = \{y^2 = x^3 + x^2\} \subset \mathbb{A}^2$, blow up the origin (ambient space):

$$\mathbb{A}^1 \simeq \tilde{Y} = \{y^2 = x^3 q + x^2\} \subset \mathrm{Tot}(\mathcal{O}(-1)_q \to \mathbb{P}^1_{x:y}) \xrightarrow{(xq,yq)} \mathbb{A}^2.$$

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Thanks!